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Consistency among trading desks

David Heath · Hyejin Ku

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Abstract We consider a bank having several trading desks, each of which trades a different class of contingent claims with each desk using a different model. We assume that the models are arbitrage-free. A practical question is whether a bank using several models can be arbitraged. Surprisingly it can happen that in some cases there must be an arbitrage. We discuss conditions under which the bank trades without offering arbitrage.

Keywords Arbitrage · Pricing operator · Countably additive measure · Martingale measure

JEL Classification G10

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D. Heath
Department of Mathematical Sciences,
Carnegie Mellon University, Pittsburgh,
PA, USA
e-mail: heath@andrew.cmu.edu

H. Ku (✉)
Department of Mathematics and Statistics,
York University, Toronto,
ON, Canada
e-mail: hku@mathstat.yorku.ca

1 Introduction

We consider a bank having several trading desks. Suppose that each desk uses its own arbitrage-free model and trades a different class of contingent claims. Can we ensure that the bank won't offer arbitrage to its counterparties? If the bank were willing to trade all securities and the models are "inconsistent", then there would be arbitrage opportunities. This is implied by the fact that there is no common pricing measure. This paper gives conditions under which the bank can be sure it does not permit arbitrage.

Some banks use the Libor market model for Libor derivatives (e.g. caps, futures etc.) and the swap market model for swap derivatives. The banks report that there seems no decent single model to explain both caps and swaption prices. This motivates the questions of whether there is "inconsistency" between the prices and whether the banks using both Libor and swap market models would possibly offer arbitrage.

Brigo and Mercurio [3] also discussed in their book that the Libor market model for pricing caps and the swap market model for pricing swaptions are "distributionally" incompatible. That is, while the forward swap rates under the swap market model are lognormally distributed, they are not lognormal under the Libor market model.

One may argue that the difference between the forward swap rates (or forward Libor rates) under the two models is not large, and this is not a problem in practice because of transaction costs or market illiquidity. However the question still remains and the possibility of arbitrage should be investigated. Although there were various works on the arbitrage-free property and the existence of a martingale measure since [6], the question of whether banks using different models can be arbitraged has not been carefully studied. Since different products motivate different models, the question we address in this paper is important for both theoretical and practical interests.

In this paper, each trading desk uses its own model and trades a linear subspace of claims. On a probability space $(\Omega, \mathcal{F}, P)$, we define *strong arbitrage* as a claim (random variable) X which satisfies $X \geq 0$ (i.e., has a nonnegative payoff a.s.), but has a negative market price $\psi(X) < 0$. We call a model with a pricing operator *weakly arbitrage-free* if there is no strong arbitrage. By a *martingale measure* (for a given pricing operator) we mean a countably additive probability measure which represents the pricing operator and is absolutely continuous with respect to P. We call a model with a pricing operator *arbitrage-free* if there exists a martingale measure.

We consider pricing operators ψ which are linear and map X's to their market prices. Ruling out trivial arbitrage permits the representation of prices by a finitely additive measure (see Theorem 3.1), and an example shows that there need not be a countably additive measure. Adding a continuity condition provides the existence of a (possibly signed) countably additive representing measure (see Theorem 4.2), but sometimes the only countably additive representing measure is a signed measure. Finally, the consistency condition guarantees the existence of a positive countably additive pricing measure.

This paper is organized as follows. In Sect. 2 we provide a brief explanation of Libor and swap market models. In Sect. 3, we provide a simple condition under which there is no strong arbitrage and there exists a finitely additive representing measure. We then show, by example, that there is "inconsistency" between Libor and swap market models when the bank is willing to trade certain types of products from both models. This is implied by the fact that there is no countably additive measure which represents both prices. In Sect. 4 and 5, we give necessary and sufficient conditions for the existence of a countably additive (signed or positive) measure which represents both initial pricing operators (and hence their natural extension).

2 Libor and swap market models

In this section, we present the Libor and swap market models and discuss the problem arising from a bank's use of different interest rate models. For the literature on the market models in detail, we refer to [7], [2], or [8].

The caps and swaptions markets are the main interest rate derivative markets. The Libor market model is popular in the caps market (for Libor derivatives), and the swap market model is popular in the swaptions market (for swap derivatives). It is known that each model is arbitrage-free. When a bank uses both models, can it be sure that there is no arbitrage opportunity?

2.1 Description of the market models

Let $\{T_1, T_2, \dots, T_n\}$ be a set of times; we assume that all times are equally spaced by δ, i.e., $T_k = T_1 + (k-1)\delta,\ k = 1, 2, \dots, n$. Let $P(t, T_k)$ be the price at time t of a T_k-maturity bond. Let $F(t, T_{k-1}, T_k)$ be the forward Libor defined by

$$F(t, T_{k-1}, T_k) = \frac{1}{\delta}\left(\frac{P(t, T_{k-1})}{P(t, T_k)} - 1\right),$$

which equals the Libor $L(T_{k-1}, T_k)$ at time T_{k-1}.

For the Libor market model, the bond price $P(t, T_k)$ is used as a numeraire and the probability measure Q^{T_k} associated with the numeraire $P(t, T_k)$ is called the forward measure. Then, the forward Libor is a martingale and is assumed to be lognormally distributed under Q^{T_k}.

Consider a T_2-maturity caplet resetting at time T_1 with a fixed rate k; the caplet pays the difference between the Libor $L(T_1, T_2)$ observed at time T_1 and the rate k. The payoff at time T_2 is

$$\delta\big(L(T_1, T_2) - k\big)^+.$$

It is well known that the price at time 0 of the caplet is given by

$$E_Q\Big[\mathrm{e}^{-\int_0^{T_2} r_s \mathrm{d}s}\delta\big(L(T_1,T_2)-k\big)^+\Big]$$

under the risk-neutral measure Q. If we use the T_2-maturity bond as a numeraire, then the forward measure (the martingale measure for this numeraire) Q^{T_2} has Radon–Nikodym derivative

$$\frac{\mathrm{d}Q^{T_2}}{\mathrm{d}Q}=\frac{\mathrm{e}^{-\int_0^{T_2} r_s \mathrm{d}s}}{P(0,T_2)}$$

(see, for instance [1, p. 191]). Therefore the price of the caplet is given by

$$P(0,T_2)E_{Q^{T_2}}\big[\delta\big(L(T_1,T_2)-k\big)^+\big].$$

Now suppose an interest rate swap starts at time T_1, the floating rate is reset equal to the Libor at times $T_1,T_2,\ldots,T_{n-1}$ and the payment times (of paying a fixed rate k and receiving the floating rates) are $T_2,\ldots,T_n$. Consider an option to enter into the swap at time T_1. Then at time T_1, the swaption is worth

$$\left(\sum_{k=2}^{n}P(T_1,T_k)\delta\big(L(T_{k-1},T_k)-k\big)\right)^+. \tag{2.1}$$

The forward swap rate is the value of the fixed rate which makes the initial swap value equal to zero. Then the forward swap rate is

$$S(t)=\frac{P(t,T_1)-P(t,T_n)}{\sum_{k=2}^{n}\delta P(t,T_k)}. \tag{2.2}$$

For the swap market model, $N^{\mathrm{S}}(t):=\sum_{k=2}^{n}\delta P(t,T_k)$ is used as a numeraire, and the probability measure Q^{S} associated with the numeraire N^{S} is called the forward swap measure. Then, the forward swap rate $S(t)$ is a martingale and is assumed to have a lognormal distribution under the measure Q^{S}. By a change of numeraire as before, the swaption price at time 0 is determined by

$$N^{\mathrm{S}}(0)E_{Q^{\mathrm{S}}}\big[\big(\mathrm{S}(T_1)-k\big)^+\big].$$

As illustrated, the above two market models use different numeraires and different pricing measures. Furthermore, computations are based on lognormality of the forward Libor rates and forward swap rates assumed by each model. That is, each of the pricing operators of two models is represented by a martingale measure, Q^{T_k} and Q^{S} respectively. We are concerned with the question of whether there is a martingale measure which represents both prices.

Remark 2.1 We remark that the reader, who worries about the boundedness of the payoffs, may consider floors and swaps (receiving fixed) instead.

2.2 Is it arbitrage-free?

Let us first consider a swaption whose underlying swap has one period, which is reset at time T_1 and pays at time T_2 $(= T_1 + \delta)$. The payoff at time T_1 the swaption is [see (2.1)]

$$P(T_1, T_2)\delta\big(L(T_1, T_2) - k\big)^+$$

with a rate k.

On the other hand, consider a caplet paying the difference between the Libor $L(T_1, T_2)$ and the rate k at time T_2. The value at time T_1 of the caplet is $P(T_1, T_2)\delta\big(L(T_1, T_2) - k\big)^+$. Therefore a swaption for single period swap can be viewed as a claim traded in both caps and swaptions markets. To price this derivative, the forward swap measure Q^{S} and the forward measure Q^{T_2} will be used in each market, respectively.

In this case, the forward swap rate is given by [see (2.2)]

$$\frac{P(t, T_1) - P(t, T_2)}{\delta P(t, T_2)} = \frac{1}{\delta}\left(\frac{P(t, T_1)}{P(t, T_2)} - 1\right)$$

which is equivalent to the forward Libor $F(t, T_1, T_2)$. Also the pricing measure Q^{S} for the swap market model coincides with the measure Q^{T_2} for the Libor market model. Two models thus give the same prices for claims traded simultaneously in both markets. Therefore there seems no trivial "cross market" arbitrage.

3 Simple condition for no strong arbitrage

On a probability space $(\Omega, \mathcal{F}, P)$ with a reference probability measure P, let L_1 and L_2 be the set of contingent claims traded by each desk. We consider L_1 and L_2 as linear subspaces of $L^\infty(\Omega, \mathcal{F}, P)$, the space of all equivalence classes of bounded real-valued functions defined on Ω. We assume we can choose a numeraire so that one of the two desks trades the claim $X = 1$ at market price $\psi(X) = 1$. Without loss of generality, assume that L_1 contains the constant claim 1.

Let ψ_1 and ψ_2 be pricing operators of L_1 and L_2, respectively (i.e., $\psi_i : L_i \to \mathbb{R}$ for $i = 1, 2$). Suppose ψ_1 and ψ_2 are linear and satisfy the property for no strong arbitrage: If $X_1 \in L_1$ and $X_1 \leq b$, then $\psi_1(X_1) \leq b$ for any constant $b \in \mathbb{R}$, and if $X_2 \in L_2$ and $X_2 \leq 0$, then $\psi_2(X_2) \leq 0$.

A trader who could trade with both desks can construct any element of $L_1 + L_2$. Thus we are concerned with the question of extending ψ_1 and ψ_2, defined on L_1 and L_2, to ψ on $L_1 + L_2$. To avoid trivial arbitrage, we assume that if the value of X_1 is always less than or equal to that of X_2, then the price for X_1 is less than or equal to the price for X_2. We note that this assumption ensures that the value of ψ_1 is equal to that of ψ_2 on $L_1 \cap L_2$. In words, the prices should be the same for contingent claims which are simultaneously traded by both desks.

Under this simple condition, we can show there is a unique extension ψ of ψ_1 and ψ_2. Moreover the Hahn–Banach theorem guarantees that there is a finitely additive probability measure μ in the dual space of $L^\infty(\Omega, \mathcal{F}, P)$ such that $\psi(X) = E_\mu[X]$.

Theorem 3.1 *Suppose that*

$$\psi_1(X_1) \leq \psi_2(X_2) \quad \text{for all } X_1 \leq X_2 \text{ a.s.} \tag{3.1}$$

where $X_1 \in L_1$ and $X_2 \in L_2$. Then there exists a unique linear map $\psi : L_1 + L_2 \to \mathbb{R}$ such that $\psi|_{L_i} = \psi_i$ for $i = 1, 2$ and ψ satisfies the property for no strong arbitrage: If $X \in L_1 + L_2$ and $X \leq b$, then $\psi(X) \leq b$ for any constant $b \in \mathbb{R}$.

Moreover, there exists a finitely additive probability measure μ which is absolutely continuous with respect to P, that is, $P(A) = 0$ implies $\mu(A) = 0$ for $A \in \mathcal{F}$, and

$$\psi(X) = E_\mu[X] \quad \text{for all } X \in L_1 + L_2.$$

Proof Define the linear map ψ on $L_1 + L_2$ such that $\psi(X) = \psi_1(X_1) + \psi_2(X_2)$ where $X = X_1 + X_2$ for $X_1 \in L_1$ and $X_2 \in L_2$. Since ψ_1 and ψ_2 are linear and $\psi_1 = \psi_2$ on $L_1 \cap L_2$, this ψ is a well-defined linear map. Also, since $\psi_1(X_1) \leq \psi_2(X_2)$ for $X_1 \leq X_2$, it follows that if $X \in L_1 + L_2$ and $X \leq b$, then $\psi(X) \leq b$.

Using the Hahn-Banach theorem (see, for example, [9, p. 47]), there is a linear functional Ψ defined on $L^\infty(\Omega, \mathcal{F}, P)$ such that $\Psi(X) = \psi(X)$ for all $X \in L_1 + L_2$. We note that the dual space of $L^\infty(\Omega, \mathcal{F}, P)$ is the space of finitely additive measures on $(\Omega, \mathcal{F}, P)$, and the property for no strong arbitrage of ψ implies that Ψ is a positive linear map on $L^\infty(\Omega, \mathcal{F}, P)$. Hence, there exists a finitely additive measure μ on $(\Omega, \mathcal{F}, P)$ for which

$$\psi(X) = E_\mu[X]$$

for all $X \in L_1 + L_2$. □

Remark 3.1 We remark that the property for no strong arbitrage (i.e., if $X \leq b$, then $\psi(X) \leq b$ for any $b \in \mathbb{R}$) also implies $|\psi(X)| \leq ||X||_\infty$.

We now consider the existence of a martingale measure for the prices of two market models. For this purpose, we rather use the following conceptual example. Let Ω be the (countably infinite) set of all possible outcomes $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots\}$. Let L_1 denote the set of claims traded in a forward Libor market, which have a constant payoff except for finitely many ω_{2k+1}'s $(k = 0, 1, 2, \ldots)$. Let L_2 denote the set of claims traded in a forward swap market, which have a constant payoff except for finitely many ω_{2k}'s $(k = 0, 1, 2, \ldots)$.

Assume that the price of a contingent claim X is determined in each market by the constant value that X takes on infinitely often.

Then, the combined set $L = L_1 + L_2$ (of claims available to a trader who can trade both types) is the set of contingent claims whose payoffs are constant for all but finitely many ω_n's, and the price of each such claim is equal to that constant. Clearly each of two pricing operators ψ_1 and ψ_2 is represented by a countably additive measure (for example, a measure giving mass 1 to the set $\{\omega_2\}$ represents ψ_1).

However, a pricing operator ψ of L, the unique extension of ψ_1 and ψ_2, cannot be represented by a countably additive measure. In fact, we have that $\psi(1_{\{\omega_n\}}) = 0$ for every ω_n where $1_{\{\omega_n\}}$ is the indicator random variable of the set $\{\omega_n\}$, because $\psi(1_{\{\omega_n\}}) = \psi_1(1_{\{\omega_n\}}) = 0$ if n is odd, and $\psi(1_{\{\omega_n\}}) = \psi_2(1_{\{\omega_n\}}) = 0$ otherwise. Suppose Q represents ψ, i.e., $\psi(X) = E_Q[X]$. If Q were countably additive, we would have $Q = 0$ since any countably additive measure assigning mass 0 to each set $\{\omega_n\}$ must be the zero measure.

As the example shows, it can happen that there is no countably additive measure which represents the pricing operators of two market models. From this observation, we conclude that even though there seems no strong arbitrage between Libor and swap market models, the bank offers an arbitrage if the bank (using two models) is willing to trade all types of securities.

We note that the above example adapts an idea of David Gilat discussed in [5].

4 Existence of a countably additive representing measure

As discussed in Sect. 3, the condition (3.1) given in Theorem 3.1 for no strong arbitrage is not sufficient for the existence of a countably additive measure which represents the pricing operator. In this section, we add a continuity condition in order to obtain a countably additive representing measure.

We consider the duality (L^∞, L^1) with the bilinear form $\langle X, f\rangle = E_P[Xf]$. Let $\mathcal{T}$ be the relative topology on $L_1 + L_2$ induced by $\sigma(L^\infty, L^1)$.

Definition 4.1 *Let ψ, defined on $L_1 + L_2$, be a linear extension of ψ_1 and ψ_2. ψ is called* **suitably continuous** *if* $\lim \psi(X_\alpha) = 0$ *for each net $\{X_\alpha\}$ in $L_1 + L_2$ which converges to* 0 *for the $\mathcal{T}$-topology.*

Now we show that if ψ is suitably continuous, there exists a countably additive measure which represents the pricing operator. In fact, this condition is necessary and sufficient for the existence of a countably additive representing measure.

Theorem 4.2 *Suppose that $\psi_1(X_1) \le \psi_2(X_2)$ for all $X_1 \le X_2$ a.s. where $X_1 \in L_1$ and $X_2 \in L_2$. Then the unique linear map $\psi : L_1 + L_2 \to \mathbb{R}$, which is an extension of ψ_1 and ψ_2, is suitably continuous if and only if there is a countably additive (not necessarily positive) measure Q which is absolutely continuous with respect to P and satisfies*

$$\psi(X) = E_Q[X]$$

for all $X \in L_1 + L_2$.

Proof Suppose that ψ satisfies the continuity condition. Consider the set

$$M = \{X \in L_1 + L_2 \mid \psi(X) = 0\}$$

and choose $X_0 \in L_1 + L_2$ such that $\psi(X_0) = 1$.

We first show that X_0 is not in $\overline{M}$ for the $\sigma(L^\infty, L^1)$-topology. To that end, suppose that X_0 is in the $\sigma(L^\infty, L^1)$-closure of M. Then there exists a net $\{X_\alpha\}$ in M converging to X_0 in the $\sigma(L^\infty, L^1)$-topology, see [4]. Since X_0 and all X_α are in $L_1 + L_2$ and $\mathcal{T}$ is the induced topology on $L_1 + L_2$, $\{X_\alpha\}$ converges to X_0 in the $\mathcal{T}$-topology. But $\lim \psi(X_\alpha)$ is not equal to 1; this contradicts the hypothesis that ψ is suitably continuous. Thus $\overline{M}$ is $\sigma(L^\infty, L^1)$-closed, and does not intersect $\{X_0\}$.

Now by the Hahn–Banach separation theorem, there exists a continuous linear functional Ψ, of the form $\Psi(X) = \langle X, f\rangle$ for all $X \in L^\infty$ and some $f \in L^1$, which separates $\overline{M}$ and $\{X_0\}$, that is, $\Psi(X_0)$ and $\Psi(M)$ are disjoint. Since $\Psi(M)$ is a linear subspace of $\mathbb{R}$, $\Psi(M) = \{0\}$ and $\Psi(X_0) = 1$ (after dividing Ψ by $\Psi(X_0)$, if necessary).

For $X \in L_1 + L_2$, we have $X - \psi(X)X_0 \in M$ since $\psi(X_0) = 1$. Then

$$\Psi(X) - \psi(X) = \Psi(X) - \psi(X)\Psi(X_0) = \Psi\big(X - \psi(X)X_0\big) = 0.$$

Thus Ψ is an extension of ψ and ψ is represented as

$$\psi(X) = E_P[Xf] = E_Q[X]$$

where Q is a countably additive (signed) measure absolutely continuous with respect to P.

Conversely, suppose that ψ is represented by $\psi(X) = E_Q[X]$ for a countably additive measure Q which is absolutely continuous with respect to P. Clearly the linear map ψ is continuous for the $\sigma(L^\infty, L^1)$-topology, so $\psi(X_\alpha)$ tends to 0 for any net $\{X_\alpha\}$ where X_α converges to 0. □

We now give an example which shows that sometimes the only countably additive representing measure must be a signed measure. For this example, we consider an extension of the one presented in Sect. 3.

Let $\Omega' = \Omega \cup \{\omega_0\}$ and extend each random variable in L_i $(i = 1, 2)$ in the following way. For each X in L_i $(i = 1, 2)$, define a random variable Y on Ω' by $Y(\omega_n) = X(\omega_n)$ for every $\omega_n \in \Omega$ and $Y(\omega_0) = -\psi(X)$, where $\psi(X)$ is the constant associated with X (as defined in the previous example). Assume that the values of ψ_i $(i = 1, 2)$ are given as before; the price of a claim is determined in each market by the constant value that the claim takes on infinitely often.

Suppose Q represents ψ. The same argument shows that any countably additive measure must assign mass 0 to the set $\{\omega_1, \omega_2, \omega_3, \ldots\}$ and hence must assign

-1 to the set $\{\omega_0\}$. Therefore if Q were countably additive, Q would be a signed measure.

5 Existence of a martingale measure

A countably additive representing measure is obtained provided the continuity condition given in Definition 4.1 holds. However, this measure is not necessarily a positive measure as required for a pricing measure. By a martingale measure (for the pricing operator) we mean a countably additive probability measure which represents a given pricing operator and is absolutely continuous with respect to P. In this section we give the consistency condition which (finally) ensures the existence of a martingale measure.

Suppose that each market is arbitrage-free and has a martingale measure. Let Q_1 and Q_2 be countably additive probability measures representing ψ_1 and ψ_2 respectively, i.e., $\psi_1(X_1) = E_{Q_1}[X_1]$ and $\psi_2(X_2) = E_{Q_2}[X_2]$ for all $X_1 \in L_1$ and $X_2 \in L_2$.

Definition 5.1 *Let ψ, defined on $L_1 + L_2$, be a linear extension of ψ_i on L_i $(i = 1, 2)$. ψ is said to satisfy the* **consistency condition** *provided there is a constant $K > 0$ such that if $E_{Q_1}[X] < 1$ and $E_{Q_2}[X] < 1$, then $\psi(X) < K$ for all $X \in L_1 + L_2$, where Q_i are probability measures representing ψ_i $(i = 1, 2)$ respectively.*

Theorem 5.2 *Suppose that $\psi_1(X_1) \leq \psi_2(X_2)$ for all $X_1 \leq X_2$ a.s. where $X_1 \in L_1$ and $X_2 \in L_2$. Then the unique linear map $\psi : L_1 + L_2 \to \mathbb{R}$, which is an extension of ψ_1 and ψ_2, satisfies the consistency condition if and only if there exists a martingale measure Q for ψ.*

Proof Suppose that ψ is represented on $L_1 + L_2$ as $\psi(X) = E_Q[X]$ for a countably additive probability measure Q which is absolutely continuous with respect to P. If we take $Q_1 = Q_2 = Q$, then the consistency condition follows with $K = 1$.

Conversely, let Q_1 and Q_2 be countably additive probability measures representing ψ_1 and ψ_2. Assume ψ satisfies the consistency condition for Q_1 and Q_2. Set

$$U = \left\{Y \in L^\infty \mid E_{Q_1}[Y] < 1,\ \ E_{Q_2}[Y] < 1\right\}.$$

Then U is a convex 0-neighborhood in L^∞ for the $\sigma\left(L^\infty(P), L^1(P)\right)$-topology. If X is in $(L_1 + L_2) \cap (U - C)$, where $C = \left\{Y \mid Y \geq 0\right\}$ is a positive cone in L^∞, then $X \leq Y$ for some Y in U, which means that $E_{Q_1}[X] < 1$ and $E_{Q_2}[X] < 1$. Therefore, the consistency condition implies that there exists a constant $K > 0$ such that $\psi(X) < K$ for all $X \in (L_1 + L_2) \cap (U - C)$.

Then, the set $\left\{X \in L_1 + L_2 \middle| \psi(X) = K\right\}$ is a linear manifold in L^∞ (i.e., a translate of a subspace of L^∞) not intersecting the open convex set $U - C$. By the separation theorem, there exists a closed hyperplane H, which can be assumed to be of the form $H = \{Y \in L^\infty | \Psi(Y) = K\}$, containing the linear manifold and not intersecting $U - C$, where Ψ is a $\sigma(L^\infty, L^1)$-continuous linear functional. Clearly, Ψ is a continuous extension of ψ and

$$\Psi(Y) = E_P[Yf]$$

for all $Y \in L^\infty$ and some $f \in L^1$. Since $0 \in U - C$, we obtain $\Psi(Y) < K$ for $Y \in U - C$. Therefore, $\Psi(Y) < K$ for $Y \in -C$, so

$$Y \in C \quad \text{implies} \quad \Psi(Y) \geq 0.$$

Hence, we have $f \geq 0$ and ψ can be represented as

$$\psi(X) = E_Q[X]$$

for a countably additive probability measure Q which is absolutely continuous with respect to P. □

References

1. Baxter, M., Rennie, A.: Financial Calculus. An Introduction to Derivative Pricing. Cambridge University Press, Cambridge (1996)
2. Brace, A., Gątarek, D. Musiela, M.: The market model of interest rate dynamics. Math. Financ. **7**, 127–147 (1997)
3. Brigo, D., Mercurio, F.: Interest Rate Models. Theory and Practice. Springer, Berlin Heidelberg New York (2001)
4. Conway, J.B.: A Course in Functional Analysis, 2nd edn. Springer, Berlin Heidelberg New York (1990)
5. Dubins, L.E., Heath, D.: On means with countably additive continuities. Proc. Am. Math. Soc. **91** (2), 270–274 (1984)
6. Harrison, J.M., Kreps, D.: Martingales and arbitrage in multiperiod securities markets. J. Econ. Theory **20**, 381–408 (1979)
7. Jamshidian, F.: Libor and swap market models and measures. Financ. Stoch. **1**, 293–330 (1997)
8. Miltersen, K., Sandmann, K., Sondermann, D.: Closed form solutions for term structure derivatives with log-normal interest rates. J. Financ. **52**, 409–430 (1997)
9. Schaefer, H.H.: Topological Vector Spaces, 2nd edn. Springer, Berlin Heidelberg New York (1999)