Stochastics: An International Journal of Probability and Stochastic Processes
Vol. 83, No. 3, June 2011, 223–231

Randomized stopping times and coherent multiperiod risk measures

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(*Received 24 April 2010; final version received 9 September 2010*)

In this paper, we derive new expectation representations of coherent multiperiod risk measures. A special feature of our representation is that it requires the use of randomized stopping times (introduced by Baxter and Chacon). Additionally, the results provide some insight into multiperiod risk measurement.

Keywords: coherent risk measures; multiperiod risk; randomized stopping times

2000 MSC code: 91B30; 91B28; 60G40

1. Introduction

This paper concerns questions related to multiperiod risk measures. Financial risk is usually quantified by a certain functional called a risk measure. If one regards the future value of a portfolio as a random variable X, a risk measure is a mapping from the set of random variables representing future portfolio values to the real numbers.

In the landmark paper, Artzner et al. [2] set out a set of properties one would want a risk measure to have and characterized the set of risk measures that possess these properties. They called such risk measures coherent and showed every coherent risk measure is represented by a set of probability measures called 'generalized scenarios'. A random variable representing future portfolio value is acceptable if its risk measure is non-positive.

Carr et al. [7] introduced valuation test measures and stress test measures instead of scenarios, and floors associated with probability measures in order to determine whether or not an opportunity is acceptable. Föllmer and Schied [11] further developed these ideas. In these papers, a random variable is considered acceptable if its expected value under each scenario measure is greater than or equal to a 'floor' associated with that measure (the floors are set to be zero for coherent risk measures).

All these papers are concerned with a single period: the risk is measured at the beginning of the period and the value of position (random loss or gain) is observed at the end of the period. It is known that a coherent risk measure is described by a set of probability distributions, in the sense that future value is acceptable if and only if every expected value under probability measures in the set is non-negative.

The notion of acceptability of a random variable was modified to some extent. Föllmer and Schied and Larsen et al. [12,16] considered the set of random variables from which it is possible, by trading, to be acceptable at the terminal date. Ku [15] defined a portfolio to be acceptable, provided there is a trading strategy (satisfying some limitations on market

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ISSN 1744-2508 print/ISSN 1744-2516 online

DOI: 10.1080/17442508.2010.523467
http://www.informaworld.com

liquidity), which, at some fixed date in the future, provided a cash-only position, (possibly) having positive future cash flows, which is acceptable as a random variable in the previous sense. However, these papers only concern final values at the end of holding period to decide its acceptability at the initial date.

Artzner et al. [3,4] turned their original static notion of coherent risk measures into a dynamic concept. The authors discussed multiperiod risks and defined coherent multiperiod risk measures on stochastic processes rather than random variables. As the investment banks would not hold static positions, risk measures should be applied to random processes, not random variables. This turns out to be quite technical and the papers worked out the problems in discrete time. The papers considered the 'risk-adjusted value' which is the negative of risk measure of a portfolio process. The authors constructed the risk-adjusted value to evolve over time satisfying Bellman's principle, and discussed a stability property of test probability measures. The authors also presented a representation result which showed such risk-adjusted value is represented by one reference probability measure on a state space and a set of positive increasing processes. In the last several years, a number of papers on dynamic risk measures have appeared, for instance [6,9,10,14,18]. This is an exciting new area of mathematical finance.

This paper concerns representations for coherent multiperiod risk measures. As mentioned, any coherent risk measure for single period arises from a set of probability measures on Ω, by computing the expected loss under each probability measure and taking the worst value. The main contribution of the paper is to provide a new characterization of multiperiod risk measure, so that one can build proper measures of risk for multiperiod. Moreover, it gives some insight into multiperiod risk measurement. For multiperiod risks, a coherent risk measure is characterized by the natural extensions of probability measures on Ω and a set of randomized stopping times, in which the additional source of randomness is involved in an interesting way. The representation theorem tells us that a portfolio process is considered acceptable for multiperiod risks if and only if the values at (randomly chosen) future times are acceptable as random variables. Also, by choosing a set of 'scenarios' on Ω along with a mixed strategy for when to observe the value, we get a coherent risk measure.

After the randomized stopping time was introduced by Baxter and Chacon [5], it has been studied by a number of authors. A randomized stopping time is a mixture of ordinary stopping times, and very useful in solving certain optimization problems or in game theory (see, for example [1] or [17]). Chalasani and Jha [8] used the randomized stopping time for American option pricing with transaction costs. As ordinary stopping times are needed in American option pricing, they represented the upper hedging price using randomized stopping times in the presence of transaction costs.

This paper is organized as follows: Section 2 describes the model, including the axioms of coherent risk measures. In Section 3, we construct a randomized stopping time and a probability measure on Ω from a probability measure on a tree in the product space $\Omega' = \{0, 1, \ldots, T\} \times \Omega$. We also present a simple example. In Section 4, we apply the results of Section 3 to obtain representation theorems for coherent multiperiod risk measures.

2. The model

Let Ω be the (finite) set of all possible events at date T. Let $\mathcal{N}_t$ be the partition of Ω consisting of the smallest events at date $t = 0, 1, \ldots, T$. These events are represented by nodes of a tree at date t as described in [3].

We note that the partition $\mathcal{N}_t (t \geq 1)$ is a refinement of the partition $\mathcal{N}_{t-1}$: we use the notation $(\underline{n}, t)$ labelled by the date t for node n on a tree, where $\underline{n}$ represents all the descendants. Then for each node $(\underline{n}, t) \in \mathcal{N}_t$, there is a node $(\underline{m}, t-1) \in \mathcal{N}_{t-1}$ satisfying $\underline{n} \subset \underline{m}$.

To denote all the nodes on the tree, we set $\mathcal{N} = \cup_{0 \leq t \leq T} \mathcal{N}_t$. Then, possible future values of position X can be viewed as a function on $\mathcal{N}$, i.e. an (adapted) process defined on the product space $\Omega' = \{0, 1, \ldots, T\} \times \Omega$, which is constant in each node of the tree. The restriction X_t of X to the partition $\mathcal{N}_t$ is also a function on Ω, and $X_t(n)$ represents the 'value' in the node n (if the event n occurs at date t).

In this paper, X is interpreted as possible future values of positions or portfolios currently held in a financial market. We call these sequences during the holding period 'value processes'. We then consider measures of risk, which describe how much capital is required for a current position to be 'acceptable'. Let $\mathcal{X}$ denote the set of all functions on $\mathcal{N}$.

Definition 2.1. A *risk measure* is a mapping from $\mathcal{X}$ into $\mathbb{R}$.

For completeness of the paper, we state axioms for measures of risk to be coherently presented in [2].

AXIOM S (subadditivity). For all $X, Y \in \mathcal{X}$,

$$\rho(X + Y) \leq \rho(X) + \rho(Y).$$

AXIOM P (positive homogeneity). For all $X \in \mathcal{X}$ and all $\alpha \geq 0$,

$$\rho(\alpha X) = \alpha \rho(X).$$

AXIOM T (translation invariance). For all $X \in \mathcal{X}$ and all real number α,

$$\rho(X + \alpha) = \rho(X) - \alpha.$$

AXIOM M (monotonicity). For all X and $Y \in \mathcal{X}$ with $X \leq Y$,

$$\rho(Y) \leq \rho(X).$$

Definition 2.2. A risk measure ρ is called *coherent* if it satisfies the above four axioms of subadditivity, positive homogeneity, translation invariance and monotonicity.

If AXIOM S and AXIOM P are replaced by AXIOM C (convexity), for all $X, Y \in \mathcal{X}$ and $0 \leq \lambda \leq 1$,

$$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y),$$

then a risk measure ρ is called convex as in [11]. (See, also [13] in which weakly coherent risk measures are considered.)

3. Randomized stopping times

In this section, we discuss a randomized stopping time and its construction process from a probability measure on $\mathcal{N}$. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_t\}$.

We extend the sample space to $\bar{\Omega} = [0, 1] \times \Omega$ and a σ-field to $\bar{\mathcal{F}} = \mathcal{B}([0, 1]) \times \mathcal{F}$ with Borel σ-algebra $\mathcal{B}([0, 1])$. We further define a filtration $\bar{\mathcal{F}}_t = \mathcal{B}([0, 1]) \times \mathcal{F}_t$ and a probability $\bar{P} = \lambda \times P$, where λ is a Lebesgue measure. Any random variable on Ω will be considered as one defined on extended sample space $\bar{\Omega} = [0, 1] \times \Omega$ in the obvious way.

DEFINITION 3.1. A map $\tau : [0,1] \times \Omega \to [0,\infty]$ is called a *randomized* $\{\mathcal{F}_t\}$*-stopping time* if τ is an $\bar{\mathcal{F}}_t$-stopping time, i.e. $\{(u,\omega) : \tau(u,\omega) \leq t\} \in \bar{\mathcal{F}}_t$ for every t.

For more details in continuous-time case, we refer to [5]. We present the following lemma, which can be obtained without too much difficulty, for the purpose of Proposition 3.3.

LEMMA 3.2. Let $([0,1], \mathcal{B}([0,1]))$ be the measurable space. Then for any $N \geq 1$, there exists an increasing family of sub σ-fields of $\mathcal{B}([0,1])$, i.e.

$$\mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_N \subset \mathcal{B}([0,1])$$

and independent random variables $X_0, X_1, \ldots, X_N$ which satisfy the following:

(1) X_i is $\mathcal{G}_i$-measurable; $(i = 0, \ldots, N)$.
(2) The distribution of X_i is uniform on $[0,1]$.

Proof. For fixed N, take independent uniform random variables $X_0, X_1, \ldots, X_N$ over [0,1]. (One can refer to random number generators for generating random variables.) Let $\mathcal{G}_i$ be the σ-algebra generated by $X_k (0 \leq k \leq i)$. □

Here, we let $\mathcal{F}_t = \bigvee_{s \leq t} \mathcal{N}_s$ and $\mathcal{F} = \bigvee_{t \leq T} \mathcal{N}_t$. Thus, $\mathcal{F}_t$ is the σ-field containing every set of $\mathcal{N}_s$ before or at time t, or the information available at time t. In other words, the sets in $\mathcal{N}_t$ are the atoms of $\mathcal{F}_t$.

PROPOSITION 3.3. For a probability measure Q on $\mathcal{N}$, there exists a pair (P, τ) consisting of a probability measure on Ω and a randomized $\{\mathcal{F}_t\}$-stopping time such that for each value process X

$$E_Q[X] = E_{\bar{P}}[X_\tau], \tag{3.1}$$

where $X_\tau = \sum_{0 \leq t \leq T} X_t 1_{\{\tau = t\}}$. Conversely, if P is a probability measure on Ω and τ is a randomized $\{\mathcal{F}_t\}$-stopping time, then there exists a probability measure Q on $\mathcal{N}$ satisfying the above equation (3.1).

Proof. Suppose that a probability Q on $\mathcal{N}$ is given. Equation (3.1) is equivalent to that for each node n,

$$Q(n) = (\lambda \times P)\{(u,\omega) : \tau(u,\omega) = t, \omega \in n\}. \tag{3.2}$$

We construct a probability measure P on Ω in the following way: first, define the function $C(\underline{n}, t)$ by

$$C(\underline{n}, t) = \sum_{n' \in \underline{n}} Q(n'),$$

which is the sum of Q-probabilities of all descendants of node n. Define the conditional probability P for each date $t (t \geq 1)$

$$P\{\omega : \omega \in n | (\underline{m}, t-1)\} = \begin{cases} \frac{C(\underline{n},t)}{C(\underline{m},t-1) - Q(m)} & \text{if } \underline{n} \subset \underline{m} \\ 0 & \text{otherwise} \end{cases},$$

and, we then define the conditional probability of n with respect to $\mathcal{N}_{t-1}$ by

$$P\{n|\mathcal{N}_{t-1}\} = \sum_{(\underline{m},t-1)\in\mathcal{N}_{t-1}} P\{\omega : \omega \in n|(\underline{m}, t-1)\}1_{\{m\}}(\omega).$$

As the convention, $P\{(n_0, 0)\} = 1$ at date 0. Then, there exists a unique probability P on $(\Omega, \mathcal{F})$.

Let N be the number of all nodes on the tree to apply Lemma 3.2. Now, we take a family of sub σ-fields of $\mathcal{B}([0, 1])$ and independent random variables at each node which satisfy the properties as in Lemma 3.2: adopting the notation $X_{(\underline{n},t)}$ for the random variable equipped with subscript representing each node $(\underline{n}, t)$,

(1) $X_{(\underline{n},t)}$ is $\mathcal{G}_t$-measurable.
(2) The distribution of $X_{(\underline{n},t)}$ is uniform on $[0, 1]$,

where $\mathcal{G}_t$ is the smallest σ-field generated by $\{X_{(\underline{n},s)}, s \le t\}$. Define a process τ (a function on $[0, 1] \times \Omega$) by

$$\tau(\cdot, \omega) = 0 \Longleftrightarrow \left(X_{(\underline{n_0},0)} \le Q(n_0)\right)$$

$$\tau(\cdot, \omega) = 1, \omega \in n \Longleftrightarrow \left(X_{(\underline{n_0},0)} > Q(n_0)\right) \quad \text{and} \quad \left(X_{(\underline{n},1)} \le \tfrac{Q(n)}{C(\underline{n},1)}\right)$$

$$\vdots$$

$$\tau(\cdot, \omega) = t, \omega \in n \Longleftrightarrow \left(X_{(\underline{m},s)} > \tfrac{Q(m)}{C(\underline{m},s)} \quad \text{for all } \underline{n} \subset \underline{m} \text{ and } s < t\right)$$
$$\text{and} \left(X_{(\underline{n},t)} \le \tfrac{Q(n)}{C(\underline{n},t)}\right).$$

Note that $(\lambda \times P)\{\tau(\cdot, \omega) \le T\} = 1$. We then observe that $\{\tau(\cdot, \omega) \le t\}$ is $\mathcal{G}_t \times \mathcal{F}_t$-measurable for each date t, thus $\overline{\mathcal{F}_t}(= \mathcal{B}([0, 1]) \times \mathcal{F}_t)$-measurable. Therefore, $\tau : [0, 1] \times \Omega \to \{0, 1, \ldots, T\}$ is a randomized $\{\mathcal{F}_t\}$-stopping time.

Finally, it can be checked that equation (3.2) holds for the probability P on Ω and a randomized $\{\mathcal{F}_t\}$-stopping time τ provided. For each node $(\underline{n}, t)$, there are nodes $(\underline{m}, t-1) \in \mathcal{N}_{t-1}$, $(\underline{l}, t-2) \in \mathcal{N}_{t-2}$, $\ldots$, repeatedly such that $\underline{n} \subset \underline{m} \subset \underline{l} \subset \cdots \subset \underline{n_0}$. Then, we have

$$(\lambda \times P)\{\tau(\cdot, \omega) = t, \omega \in n\} = \frac{Q(n)}{C(\underline{n}, t)} P\{\omega \in n|(\underline{m}, t-1)\}$$
$$\left(1 - \frac{Q(m)}{C(\underline{m}, t-1)}\right) P\{m|(\underline{l}, t-2)\} \cdots \left(1 - \frac{Q(n_0)}{C(\underline{n_0}, 0)}\right) P\{\omega \in n_0\}.$$

The above equation becomes

$$\frac{Q(n)}{C(\underline{n}, t)} \frac{C(\underline{n}, t)}{C(\underline{m}, t-1) - Q(m)} \left(1 - \frac{Q(m)}{C(\underline{m}, t-1)}\right)$$
$$\frac{C(\underline{m}, t-1)}{C(\underline{l}, t-2) - Q(l)} \cdots \left(1 - \frac{Q(n_0)}{C(\underline{n_0}, 0)}\right),$$

which equals $Q(n)$.

Conversely, suppose P is a probability measure on Ω and τ is a randomized $\{\mathcal{F}_t\}$-stopping time. We simply define the values of Q on $\mathcal{N}$ by the formula

$$Q(n) = (\lambda \times P)\{(u, \omega) : \tau(u, \omega) = t, \omega \in n\}.$$

Then,

$$\sum_{0 \le t \le T} \sum_{n \in \mathcal{N}_t} Q(n) = \sum_{0 \le t \le T} (\lambda \times P)\{\tau(\cdot, \omega) = t, \omega \in n\} = 1.$$

Therefore, Q is a probability measure on $\mathcal{N}$. □

We provide an example to illustrate the construction of P and τ in the simplest setting. The tree in general does not of course have to be binomial.

Example 3.4. Let $\Omega = \{HH, HT, TH, TT\}$. Suppose that a probability measure Q on a tree is given to each node: (Figure 1).

Then, the conditional probabilities on each segment are defined as follows: (Figure 2).

The stopping probabilities at each node are also defined (Figure 3) as follows.

At time 0, it stops with probability 0.3. If it was not stopped at time 0 and got to node H at time 1, then it would stop there with probability 0.1 divided by 0.25, i.e. in the proportion of Q-weight to all future weights. (For the value process, if the process has not been observed at time 0 and got to node H, the value process would be observed there with probability (0.1/0.25.) Thus the stopping probability at node H is $(1 - 0.3)(0.1/0.25) = 0.28$. Note that the sum of stopping probabilities along any path is 1. Therefore, the randomized stopping time τ is defined on $[0, 1] \times \Omega$. For instance, given $\omega = HH$, $\tau(u, \omega)$ takes the values of 0 for $0 \le u \le 0.3$, 1 for $0.3 < u \le 0.58$ and 2 for $0.58 < u \le 1$, respectively.

By our construction of P and τ, the (natural extension of) probability that true outcome is HH and stops at time 2 will be

$$\begin{aligned}\bar{P}\{\tau(u, HH) = 2\} &= (1 - 0.3)\frac{0.25}{0.7}\left(1 - \frac{0.1}{0.25}\right)\frac{0.1}{0.15} = 0.1 \\ &= Q(HH, 2).\end{aligned}$$

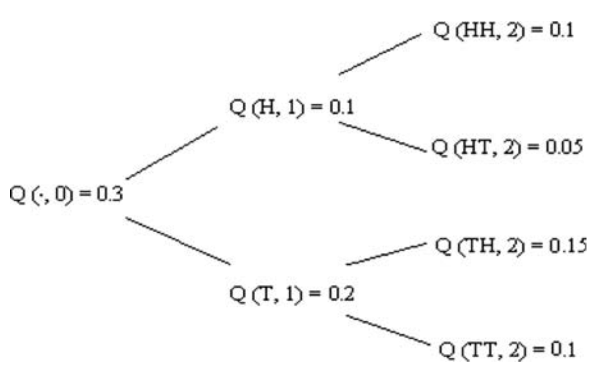

Figure 1. Probability Q on a tree.

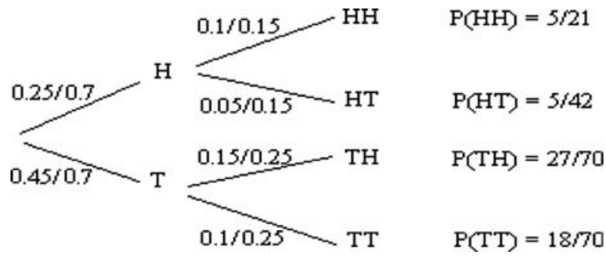

Figure 2. Conditional probability P.

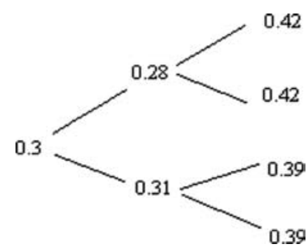

Figure 3. Randomized stopping time τ.

In the same way, $\bar{P}\{\tau(u,\omega)=t\}=Q(\omega,t)$ for every ω and t.

Remark 3.5. In Figure 3, the sum of the values along any path is 1. For an ordinary stopping time, the value of 1 appears at only one node in each path and the value of 0 at all other nodes. While ordinary stopping times specify exactly one time point to stop on each path, randomized stopping times are generalizations to add the additional 'randomness' to stopping times on the base sample space. In other words (for financial intuition), the value process may be observed at any time, which is randomly chosen by the regulator.

4. Representation of a coherent multiperiod risk measure

In this section, we consider risk measures on value processes, the sequence of unknown future values at intermediate dates of a portfolio process. In [3,4], the authors considered the 'risk-adjusted value' which is the negative of risk measure of a portfolio process and showed for a multiperiod risk measure that satisfies the four coherence axioms (given in Section 2), the risk-adjusted value of each value process is given by a set of test probabilities on product space Ω' as follows:

For each coherent risk measure, there exists a family $\mathcal{Q}$ of probability measures Q on Ω' such that for each value process X, its risk-adjusted value $\pi(X)$ at date 0 is given by

$$\pi(X)=\inf_{Q\in\mathcal{Q}}\{E_Q[X]\}.$$

By the duality argument as in the single-period case, the above result has been obtained. Also, one can build a multiperiod risk measure from the selection of test probabilities on the product space. However, one feels more comfortable with working for 'scenarios' on Ω, rather than generating the test probabilities on the product space Ω'.

Artzner et al. [3,4] also had the representation result by a collection of positive increasing processes under a reference probability as follows:

For each coherent risk measure, there is a set $\mathcal{A}$ of positive increasing processes A with $E_P[A_T] = 1$ such that for each value process X, its risk-adjusted value $\pi(X)$ at date 0 is given by

$$\pi(X) = \inf_{A \in \mathcal{A}} E_P \left[\sum_{0 \le t \le T} X_t (A_t - A_{t-1}) \right].$$

The increments $(A_t - A_{t-1})$ are the density with respect to the reference probability on the product space Ω'. But, a set of increasing processes $\mathcal{A}$ does not seem to be easily applicable in practice.

In this section, we present the representation result for a coherent multiperiod risk measure by probability measures on a sample space Ω and randomized stopping times. Proposition 3.3 is applied to get the following.

PROPOSITION 4.1. If a multiperiod risk measure ρ is coherent, then there exists a family $\{(\mathcal{P}, \mathcal{T})\}$ of pairs consisting of probability measures on Ω and randomized stopping times such that for each value process X

$$\rho(X) = - \inf_{(P,\tau) \in \{(\mathcal{P},\mathcal{T})\}} \{E_{\bar{P}}[X_\tau]\},$$

where $X_\tau = \sum_{0 \le t \le T} X_t 1_{\{\tau = t\}}$. Conversely, from a set of distributions of a sample space Ω along with randomized stopping times, we get a coherent risk measure for multiperiod risks.

From the representation theorem, a risk measure is determined by 'generalized scenarios' on base sample space Ω and a mixed strategy for when to observe the value. Then, the worst value of the (coherent) risk measure is considered for multiperiod risks. Since the time at which the regulator is coming to check the values is uncertain, the value must be acceptable no matter when the regulator is coming.

Since ordinary stopping times are extreme points of randomized stopping times, we note that a collection of pairs of probability measures on Ω and ordinary stopping times generates a coherent multiperiod risk measure. However, not every coherent risk measure can be represented by ordinary stopping times along with probability measures on Ω.

Remark 4.2. One could consider more general acceptance sets than convex cones, as was done in [11]. If a multiperiod risk measure ρ is convex, then there exists a family $\{(\mathcal{P}, \mathcal{T})\}$ of pairs consisting of probability measures on Ω and randomized stopping times and corresponding constants $C_{(P,\tau)}$ such that

$$\rho(X) = - \inf_{(P,\tau) \in \{(\mathcal{P},\mathcal{T})\}} \{E_{\bar{P}}[X_\tau] + C_{(P,\tau)}\},$$

for each value process X.

5. Conclusion

Measurement and management of financial risks are important topics in mathematical finance, and one of the critical steps is to construct proper measures of risk. The present paper has derived a new characterization for coherent multiperiod risk measures.

Due to this contribution, one can build a coherent risk measure (for multiperiod risks) by choosing 'generalized scenarios' on base sample space Ω and a mixed strategy for when to observe the value. For multiperiod coherent risk measures, it is described by a set of probability measures on Ω and a set of stopping times, in which the additional source of randomness is involved in an interesting way. This can be done by the fact that the expected value under a probability measure on a product space Ω' can be represented as the expected value under a probability measure on a sample space Ω at date that is randomly determined by a randomized stopping time. In words, the presence of randomized stopping time implies the randomness for the time the value process is being observed (i.e. a mixed strategy for when to observe the value). Therefore, a portfolio process is considered 'acceptable' for multiperiod risks if and only if the values at (randomly chosen) future times are acceptable as random variables.

Acknowledgement

This research has been supported by the NSERC discovery grant no. 504316.

References

[1] D. Aldous and P. Diaconis, *Shuffling cards and stopping times*, Amer. Math. Monthly 93 (1986), pp. 333–348.
[2] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath, *Coherent measures of risk*, Math. Finance 9 (1999), pp. 203–228.
[3] P. Artzner, F. Delbaen, J.-M. Eber, D. Heath, and H. Ku, *Coherent multiperiod risk measurement*, manuscript (2002). Available at http://www.math.ethz.ch/delbaen/ftp/preprints/MULTIPERIOD-3-02.pdf
[4] P. Artzner, F. Delbaen, J.-M. Eber, D. Heath, and H. Ku, *Coherent multiperiod risk adjusted values and Bellman's principle*, Ann. Oper. Res. 152 (2007), pp. 5–22.
[5] J.R. Baxter and R.V. Chacon, *Compactness of stopping times*, Z. Volkskunde. 40 (1977), pp. 169–181.
[6] J. Bion-Nadal, *Bid-ask dynamic pricing in financial markets with transaction costs and liquidity risk*, J. Math. Econom. 45 (2009), pp. 738–750.
[7] P. Carr, H. Geman, and D. Madan, *Pricing and hedging in incomplete markets*, J. Finance Econom. 62 (2001), pp. 131–167.
[8] P. Chalasani and S. Jha, *Randomized stopping times and American option pricing with transaction costs*, Math. Finance 11 (2001), pp. 33–77.
[9] P. Cheridito, F. Delbaen, and M. Kupper, *Coherent and convex monetary risk measures for unbounded cadlag processes*, Finance Stoch. 9 (2006), pp. 369–387.
[10] S. Drapeau and M. Kupper, *Risk preferences and their robust representation*, preprint (2010). Available at http://www.mathematik.hu-berlin.de/~kupper/DKRPRR0710.pdf.
[11] H. Föllmer and A. Schied, *Stochastic Finance*, de Gruyter, Berlin, 2002a.
[12] H. Föllmer and A. Schied, *Convex measures of risk and trading constraints*, Finance Stoch. 6 (2002b), pp. 429–447.
[13] D. Heath and H. Ku, *Pareto equilibria with coherent measures of risk*, Math. Finance 14 (2004), pp. 163–172.
[14] A. Jobert and L.C.G. Rogers, *Valuations and dynamic convex risk measures*, Math. Finance 18 (2008), pp. 1–22.
[15] H. Ku, *Liquidity risk with coherent risk measures*, Appl. Math. Finance 13 (2006), pp. 131–141.
[16] K. Larsen, T. Pirvu, S. Shreve, and R. Tutuncu, *Satisfying convex risk limits by trading*, Finance Stoch. 9 (2005), pp. 177–195.
[17] P. Neumann, D. Ramsey, and K. Szajowski, *Randomized stopping times in Dynkin games*, ZAMM-Z Angew. Math. Mech. 82 (2002), pp. 811–819.
[18] F. Riedel, *Dynamic coherent risk measures*, Stochastic Process. Appl. 112(2) (2004), pp. 185–200.