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# Valuation of American partial barrier options

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**Abstract** This paper concerns barrier options of American type where the underlying asset price is monitored for barrier hits during a part of the option's lifetime. Analytic valuation formulas of the American partial barrier options are provided as the finite sum of bivariate normal distribution functions. This approximation method is based on barrier options along with constant early exercise policies. In addition, numerical results are given to show the accuracy of the approximating price. Our explicit formulas provide a very tight lower bound for the option values, and moreover, this method is superior in speed and its simplicity.

**Keywords** Partial barrier option · American option · Hitting time · Barrier approximation

**JEL Classification** G13 · C65

## 1 Introduction

Barrier options are widely traded in over-the-counter markets because they are more flexible and cheaper than vanilla options. These options either cease to exist or come into existence when some pre-specified asset price barrier is hit during the option's life. Merton (1973) has derived a down-and-out call price by solving the corresponding partial differential equation with some boundary conditions. Rubinstein and Reiner (1991) published closed form pricing formulas for various types of single barrier

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options. Rich (1994) also provided a mathematical framework to value barrier options. Due to their popularity in a market, more complicated structures of barrier options have been studied by a number of authors. Kunitomo and Ikeda (1992) derived a pricing formula for double barrier options with curved boundaries as the sum of an infinite series. Geman and Yor (1996) followed a probabilistic approach to derive the Laplace transform of the double barrier option price. In these papers, the underlying asset price is monitored for barrier hits or crossings during the entire life of the option.

On the other hand, Heynen and Kat (1994) studied partial barrier options where the underlying price is monitored during only part of the option's lifetime. Partial barrier options have two classes. One is forward starting barrier options where the barrier appears at a fixed date strictly after the option's initial starting date. The other is early ending barrier options where the barrier disappears at a specified date strictly before the expiry date. They can be applied for various types of options according to the clients' needs as controlling the starting or ending time of the monitoring period. Also, they can be used as components to synthetically create other types of exotic options. Heynen and Kat (1994) gave valuation formulas for partial barrier options in terms of bivariate normal distribution functions. As a natural variation on the partial barrier structure, window barrier options have become popular with investors, particularly in foreign exchange markets. For a window barrier option, a monitoring period for the barrier commences at the forward starting date and terminates at the early ending date. (We refer to Guillaume 2003.)

In the case of American options which give their holders the additional flexibility of early exercise, an exact and closed-form pricing solution has not existed because the option price and the early exercise boundary must be determined simultaneously. Consequently, the literature of American options has proposed only numerical solution methods and analytical approximations.

The numerical methods include the finite difference method by Brennan and Schwartz (1977) and Parkinson (1977) and the binomial model of Cox et al. (1979). These numerical methods are quite flexible and simple to implement. However, even after employing enhancement techniques such as control variates or convergence extrapolation, they are very time consuming.

There are many approximation schemes developed to reduce this time consuming task. Johnson (1983) expressed the put value as an approximate function of its parameters. Geske and Johnson (1984) approximated the American option price through an infinite series of multivariate normal distribution functions. Barone-Adesi and Whaley (1987) used Merton's (1973) solution for perpetual American options and the quadratic method of MacMillan (1986). Despite its high efficiency and the accuracy improvements, this method is not convergent because there is no control parameter to adjust to improve the accuracy.

Longstaff and Schwartz (2001) adapted Monte Carlo simulation methods to deal with the American put problem. They addressed the optimal stopping problem in a Monte Carlo framework by comparing the conditional expected value of continuing with the value of immediate exercise if the option is currently in the money. Sullivan (2000) approximated the option value function through Chebyshev polynomials and employed a Gaussian quadrature integration scheme at each discrete exercise date. Although the speed and accuracy of the proposed numerical approximation can be enhanced via the Richardson extrapolation, its convergence properties are still unknown.

Kim (1990), Jacka (1991), and Carr et al. (1992) obtained an analytic integral-form solution for American options where the formulas represent the early premium of an American option as an integral which has the early exercise boundary. Broadie and Detemple (1996) provided tight lower and upper bounds for American call prices based on the assumption that the early exercise boundary is a constant. Ju (1998) approximated the early exercise boundary as a multipiece exponential function and substitute it by the early exercise premium integral, derived by Kim (1990), to price American options. Ingersoll (1998) described another approximation method of American options based on barrier options: The exercise policy is approximated by a simple class of functions, and the best policy within that class is selected by standard optimization techniques. The advantages of this method are its simplicity and speed, even when used in general-purpose computer programs such as spreadsheets. Concretely, he dealt with a constant barrier approximation and an exponential barrier approximation for American put. Chung et al. (2010) derived the essential formulae for solving the lower bound and the optimal exercise boundary.

For the American barrier option problem, Gao et al. (2000) suggested an approximation method for American barrier options. They applied the approximation techniques of a standard American option to an American barrier option, and proposed two approximation methods using Huang et al. (1996) and Ju (1998) to approximate an American barrier's exercise boundary. Dai and Kwok (2004) provided an analytic formula for knock-in options and showed that the in-out barrier parity relationship for American barrier options could not be obtained unlike the case of European barrier options. Ingersoll (1998) presented American up-and-in put price by an approximation method based on barrier options using constant and exponential exercise policies.

This paper concerns the barrier option of American type where the barrier appears at a fixed date strictly after the option's initial starting date. To the best of our knowledge, the literature of American exotic option suggests no approximation formula for American partial barrier options. Moreover, the numerical methods such as Monte Carlo method and Lattice method for these options demand much time. This paper extends the approximation method of American barrier option suggested in Ingersoll (1998) to the case of partial barrier options of American type. The constant functions are considered for early exercise boundaries. By our method, American partial barrier option can be valued in a simple and speedy way.

This article is organized as follows. Section 2 presents a review of valuing American barrier option using barrier derivatives. Section 3 provides the analytic approximation of American partial barrier option. This section is divided into two subsections. The first subsection covers the case that up-barrier is greater than or equal to strike price. The second one presents valuation formulas for the digitals when up-barrier is less than strike price. Finally, Sect. 4 provides the conclusion.

## 2 American barrier option using barrier derivatives: a review

In this section, we present a brief review of the valuation for American barrier option using barrier derivatives described in Ingersoll (1998). This method provides a good approximation to the option price with the advantages of its simplicity and speed.

Let $r$ be the risk-free interest rate, $q$ be a dividend rate, and $\sigma > 0$ be a constant. We assume the price of the underlying asset $S$ follows a geometric Brownian motion

$$S_t = S_0 \exp(\mu t + \sigma W_t)$$

where $\mu = r - q - \frac{\sigma^2}{2}$ and $W_t$ is a standard Brownian motion under the risk-neutral probability $P$.

An American up-and-in put option will be exercised when it is sufficiently in the money, but only after the stock price has risen to the knock-in barrier (or instrike). To value this contract, it will be convenient to introduce the following digitals: Let $\mathcal{D}(S, t\,;\,A)$ be the value at time $t$ of receiving one dollar at time $T$ if and only if the event $A$ occurs, and $\mathcal{DS}(S, t\,;\,A)$ be the value at time $t$ of receiving one share of stock at time $T$ if and only if the event $A$ occurs. The $\mathcal{D}$ is said to be a digital or binary option and the $\mathcal{DS}$ is said to be a digital share. The quantity $\mathcal{E}(S, t, K_\tau\,;\,A)$ denotes the value at time $t$ of payment $X - K_\tau$ at the first time $\tau$ that the stock price $S$ hits the barrier $K_\tau$ provided the event $A$ occurs before time $T$, where $X$ is a strike price. The $\mathcal{E}$ is said to be a first-touch digital.

Consider an American up-and-in put expiring $T$ with strike price $X$. Let us denote by $U$ the up-barrier and by $K_t^*$ the optimal exercise policy. Let $\tau_{B_1}$ denote the first time the stock price is equal to $B_1$ and $\tau_{B_1 B_2}$ denote the first time after $\tau_{B_1}$ that the stock price is equal to $B_2$.

Let $A_1 = \{t < \tau_U < T,\ \tau_{UK^*} > T,\ S_T < X\}$ be the event of exercise at maturity under the optimal policy, and $A_2 = \{t < \tau_U, \tau_{UK^*} < T\}$ be the event of early exercise under the optimal policy. Then the value of the up-and-in put can be written as

$$UIP = X \cdot \mathcal{D}(S, t\,;\,A_1) - \mathcal{DS}(S, t\,;\,A_1) + \mathcal{E}(S, t, K_t^*\,;\,A_2)$$

The barrier approximation for this put takes the maximum value within a class of restricted policies. For example, for constant exercise policies $k$,

$$UIP \geq UIP_{\text{const}} = \max_k \left[X \cdot \mathcal{D}(S, t\,;\,A_3) - \mathcal{S}(S, t\,;\,A_3) + \mathcal{E}(S, t, k\,;\,A_4)\right]$$

where $A_3 = \{t < \tau_U < T,\ \tau_{Uk} > T,\ S_T < X\}$, $A_4 = \{t < \tau_U, \tau_{Uk} < T\}$, and $\tau_{Uk}$ is the first time the stock price hits the constant policy barrier $k$ after hitting the barrier $U$. The values for these digitals are given by

$$\begin{aligned}\mathcal{D}(S, t\,;\,A_3) = e^{-r(T-t)} &\left\{\left(\frac{U}{S_t}\right)^{\frac{2\mu}{\sigma^2}} \left[N\left(h_1\left(\frac{U^2}{S_t k}\right)\right) - N\left(h_1\left(\frac{U^2}{S_t X}\right)\right)\right]\right. \\ &\left. + \left(\frac{k}{U}\right)^{\frac{2\mu}{\sigma^2}} \left[N\left(h_1\left(\frac{S_t k^2}{U^2 X}\right)\right) - N\left(h_1\left(\frac{S_t k}{U^2}\right)\right)\right]\right\}\end{aligned}$$

$$\mathcal{DS}(S,t\,;A_3)=S_te^{-q(T-t)}\left\{\left(\frac{U}{S_t}\right)^{\frac{2\overline{\mu}}{\sigma^2}}\left[N\left(h_2\left(\frac{U^2}{S_tk}\right)\right)-N\left(h_2\left(\frac{U^2}{S_tX}\right)\right)\right]\right.$$
$$\left.+\left(\frac{k}{U}\right)^{\frac{2\overline{\mu}}{\sigma^2}}\left[N\left(h_2\left(\frac{S_tk^2}{U^2X}\right)\right)-N\left(h_2\left(\frac{S_tk}{U^2}\right)\right)\right]\right\}$$
$$\mathcal{E}(S,t,k\,;A_4)=(X-k)$$
$$\times\left[\left(\frac{k}{S_t}\right)^{b-\beta}\left(\frac{k}{U}\right)^{2\beta}N\left(g_1\left(\frac{S_tk}{U^2}\right)\right)\right.$$
$$\left.+\left(\frac{k}{S_t}\right)^{b+\beta}\left(\frac{U}{k}\right)^{2\beta}N\left(-g_1\left(\frac{U^2}{S_tk}\right)\right)\right]$$

where $N$ is the standard normal distribution function,

$$h_1(z)=\frac{\ln z+\mu(T-t)}{\sigma\sqrt{T-t}},\quad h_2(z)=\frac{\ln z+\overline{\mu}(T-t)}{\sigma\sqrt{T-t}},\quad g_1(z)=\frac{\ln z+\beta\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$
$$\mu=r-q-\frac{1}{2}\sigma^2,\quad \overline{\mu}=r-q+\frac{1}{2}\sigma^2,\quad b=\frac{\mu}{\sigma^2},\quad\text{and}\quad \beta=\sqrt{b^2+\frac{2r}{\sigma^2}}\,.$$

## 3 Analytic approximation for American partial barrier options

In this section, we consider the partial barrier option of American type. American options give their holders the flexibility of early exercise. An American up-and-in put option can be exercised before the expiration time when it is in the money, but only after the stock price rises above the knock-in barrier. We consider the up-and-in put where the barrier appears at a specified time $T_1$ strictly after the option's initiation. That is, if the underlying asset price hits the up-barrier over the time period between $T_1$ and expiration $T$, then the put option can be exercised before or at time $T$ with strike price $X$. If the asset price never crosses the up-barrier between $T_1$ and expiration $T$, this option pays off zero.

In order to obtain the approximation to valuing American partial barrier option using barrier derivatives under exercise policies, we use the digital opions $\mathcal{D}(S,t\,;A)$, $\mathcal{DS}(S,t\,;A)$ and $\mathcal{E}(S,t,K_\tau\,;A)$ for $t<T_1$ defined in Sect. 2. We denote by $\tau_{U(T_1)}$ the first time that the stock price reaches the barrier $U$ after time $T_1$. For $\tau_{Uk(T_1)}$, it is the first time that the stock price falls to the exercise policy $k$ after $\tau_{U(T_1)}$. Let $K^*$ denote the optimal exercise policy. Let $A_5=\{\tau_{U(T_1)}<T,\ \tau_{UK^*(T_1)}>T,\ S_T<X\}$ be the event of exercise at maturity under the optimal policy, and $A_6=\{\tau_{UK^*(T_1)}<T\}$ be the event of early exercise under the optimal policy.

Then the value of this partial up-and-in put is written as

$$PUIP=X\cdot\mathcal{D}(S,t\,;A_5)-\mathcal{DS}(S,t\,;A_5)+\mathcal{E}(S,t,K_t^*\,;A_6)$$

For the barrier approximation of this option, we consider a class of all constant exercise policies. We let $A_7=\{\tau_{U(T_1)}<T,\ \tau_{Uk(T_1)}>T,\ S_T<X\}$ be the event of

exercise at maturity under a constant policy $k$, and $A_8 = \{\tau_{Uk(T_1)} < T\}$ be the event of early exercise under policy $k$. Then we can express the option price as

$$PUIP_{\text{const}} = \max_{k \in \mathcal{K}_c} [X \cdot \mathcal{D}(S, t\ ;\ A_7) - \mathcal{DS}(S, t\ ;\ A_7) + \mathcal{E}(S, t, k\ ;\ A_8)] \tag{3.1}$$

If the set of policies considered contains all continuous functions, then the resulting put value will be exact. Since the set $\mathcal{K}_c$ is the set of all constant functions, then the resulting value will be an approximation providing a (very tight) lower bound to the put price.

We first present a useful Lemma to calculate the values $\mathcal{D}$, $\mathcal{DS}$, and $\mathcal{E}$ of digital, digital share, and first-touch digital. We recall that the standard normal density function and distribution function

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \ \text{ and } \ N(x) = \int_{-\infty}^{x} n(t)dt,$$

and the bivariate standard normal distribution function

$$N_2(a, b; \rho) = \int_{-\infty}^{a} \int_{-\infty}^{b} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right) dxdy$$

where $\rho$ is the coefficient of correlation.

**Lemma 3.1** *For any real $a, \alpha, \beta, \gamma$, and $\delta$,*

$$\int_{-\infty}^{a} \frac{1}{\delta} n\left(\frac{t-\gamma}{\delta}\right) N(\alpha + \beta t)dt = N_2\left(\frac{a-\gamma}{\delta}, \frac{\alpha + \beta\gamma}{\sqrt{1+\beta^2\delta^2}}\ ;\ \frac{-\beta\delta}{\sqrt{1+\beta^2\delta^2}}\right)$$

$$\int_{a}^{\infty} \frac{1}{\delta} n\left(\frac{t-\gamma}{\delta}\right) N(\alpha + \beta t)dt = N_2\left(\frac{\gamma - a}{\delta}, \frac{\alpha + \beta\gamma}{\sqrt{1+\beta^2\delta^2}}\ ;\ \frac{\beta\delta}{\sqrt{1+\beta^2\delta^2}}\right)$$

*Proof* Letting $u = \frac{t-\gamma}{\delta}$,

$$\int_{-\infty}^{a} \frac{1}{\delta} n\left(\frac{t-\gamma}{\delta}\right) N(\alpha + \beta t)dt = \int_{-\infty}^{\frac{a-\gamma}{\delta}} \int_{-\infty}^{\alpha+\beta(\delta u+\gamma)} \frac{1}{2\pi} e^{-\frac{(u^2+v^2)}{2}} dvdu.$$

Change the variables and define a coefficient of correlation $\rho$ as follows:

$$x = u, \quad y = \frac{v - \beta\delta u}{\sqrt{1+\beta^2\delta^2}}, \quad \rho = \frac{-\beta\delta}{\sqrt{1+\beta^2\delta^2}}.$$

Then

$$
\begin{aligned}
&\int_{-\infty}^{a} \frac{1}{\delta} n\left(\frac{t-\gamma}{\delta}\right) N(\alpha+\beta t)dt \\
&= \int_{-\infty}^{\frac{a-\gamma}{\delta}} \int_{-\infty}^{\frac{\alpha+\beta\gamma}{\sqrt{1+\beta^2\delta^2}}} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right) dxdy \\
&= N_2\left(\frac{a-\gamma}{\delta}, \frac{\alpha+\beta\gamma}{\sqrt{1+\beta^2\delta^2}}; \frac{-\beta\delta}{\sqrt{1+\beta^2\delta^2}}\right).
\end{aligned}
$$

For the integral

$$
\int_{a}^{\infty} \frac{1}{\delta} n\left(\frac{t-\gamma}{\delta}\right) N(\alpha+\beta t)dt,
$$

we can get the above result by a similar method. □

Let us introduce a process $X_t = \frac{1}{\sigma}\ln\left(\frac{S_t}{S_0}\right)$. Then $X_t$ is a Brownian motion with drift $\frac{\mu}{\sigma}$. Define $\tau_{u(T_1)}$ and $\tau_{ul(T_1)}$ by stopping times for this process defined as the first time that $X_t = u > X_0$ after time $T_1$ and the first time after $\tau_{u(T_1)}$ that $X_t = l < u$, respectively.

**Lemma 3.2** *For $x \geq l$, the probability that the process $X_t$ crosses $u$ after time $T_1$, and then hits $l$ before expiration $T$, and $X_T$ is greater than $x$ is*

$$
\begin{aligned}
&P(\tau_{ul(T_1)} \leq T,\ X_T > x \mid X_0 = 0) \\
&\quad = \exp\left(\frac{2\mu}{\sigma}(l-u)\right) N_2\left(\frac{u-\frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{2l-2u-x+\frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right) \\
&\qquad + \exp\left(\frac{2\mu l}{\sigma}\right) N_2\left(\frac{-u-\frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{2l-x+\frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right)
\end{aligned}
$$

*Proof*

$$
\begin{aligned}
&P(\tau_{ul(T_1)} \leq T,\ X_T > x \mid X_0 = 0) \\
&\quad = P(X_{T_1} < u,\ \tau_{ul(T_1)} \leq T,\ X_T > x \mid X_0 = 0) \\
&\qquad + P(X_{T_1} \geq u,\ \tau_{ul(T_1)} \leq T,\ X_T > x \mid X_0 = 0)
\end{aligned}
\tag{3.2}
$$

Since $u > l$, $\left\{X_{T_1} \geq u,\ \tau_{ul(T_1)} \leq T\right\}$ is equivalent to $\left\{X_{T_1} \geq u,\ \tau_{l(T_1)} \leq T\right\}$. Then

$$\begin{aligned}
&P(\tau_{ul(T_1)} \leq T,\ X_T > x \mid X_0 = 0) \\
&\quad = P(X_{T_1} < u,\ \tau_{ul(T_1)} \leq T,\ X_T > x \mid X_0 = 0) \\
&\qquad + P(X_{T_1} \geq u,\ \tau_{l(T_1)} \leq T,\ X_T > x \mid X_0 = 0) \\
&\quad = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_1 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} P(\tau_{ul(T_1)} \leq T,\ X_T > x \mid X_{T_1} = x_1)dx_1 \\
&\qquad + \int_{u}^{\infty} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_2 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} P(\tau_{l(T_1)} \leq T,\ X_T > x \mid X_{T_1} = x_2)dx_2
\end{aligned}$$

Using Lemma 1 in the Appendix of Ingersoll (1998), we have

$$\begin{aligned}
&P(\tau_{ul(T_1)} \leq T,\ X_T > x \mid X_0 = 0) \\
&= \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_1 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} e^{\frac{2\mu}{\sigma}(l-u)} N\left(\frac{x_1 + 2l - 2u - x + \frac{\mu}{\sigma}(T - T_1)}{\sqrt{T - T_1}}\right) dx_1 \\
&\quad + \int_{u}^{\infty} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_2 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} e^{\frac{2\mu}{\sigma}(l-x_2)} N\left(\frac{2l - x_2 - x + \frac{\mu}{\sigma}(T - T_1)}{\sqrt{T - T_1}}\right) dx_2 \\
&= e^{\frac{2\mu}{\sigma}(l-u)} \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_1 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} N\left(\frac{x_1 + 2l - 2u - x + \frac{\mu}{\sigma}(T - T_1)}{\sqrt{T - T_1}}\right) dx_1 \\
&\quad + e^{\frac{2\mu l}{\sigma}} \int_{u}^{\infty} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_2 + \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} N\left(\frac{2l - x_2 - x + \frac{\mu}{\sigma}(T - T_1)}{\sqrt{T - T_1}}\right) dx_2
\end{aligned}$$

Applying Lemma 3.1, we obtain

$$\begin{aligned}
&P(\tau_{ul(T_1)} \leq T,\ X_T > x \mid X_0 = 0) \\
&\quad = \exp\left(\frac{2\mu}{\sigma}(l - u)\right) N_2\left(\frac{u - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{2l - 2u - x + \frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right) \\
&\qquad + \exp\left(\frac{2\mu l}{\sigma}\right) N_2\left(\frac{-u - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{2l - x + \frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right)
\end{aligned}$$

□

**Lemma 3.3** *The probability that the process $X_t$ crosses $u$ after time $T_1$, and then falls below $l$ before time $T$ is*

$$\begin{aligned}
&P(\tau_{ul(T_1)} \le T \mid X_0 = 0) \\
&\quad = \exp\left(\frac{2\mu}{\sigma}(l-u)\right) N_2\left(\frac{u - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{l - 2u + \frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right) \\
&\qquad + \exp\left(\frac{2\mu l}{\sigma}\right) N_2\left(\frac{-u - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{l + \frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right) \\
&\qquad + \exp\left(\frac{2\mu u}{\sigma}\right) N_2\left(\frac{u + \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{l - 2u - \frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right) \\
&\qquad + N_2\left(\frac{-u + \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{l - \frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right)
\end{aligned}$$

*Proof* We note that

$$\begin{aligned}
&P(\tau_{ul(T_1)} \le T \mid X_0 = 0) \\
&\quad = P(\tau_{ul(T_1)} \le T, X_T > l \mid X_0 = 0) + P(\tau_{u(T_1)} \le T, X_T \le l \mid X_0 = 0) \quad (3.3)
\end{aligned}$$

since $\left\{\tau_{ul(T_1)} \le T, X_T \le l\right\} = \left\{\tau_{u(T_1)} \le T, X_T \le l\right\}$. The first probability of (3.3) is given by Lemma 3.2 with $x = l$ and the second one can be calculated by a similar method to the proof of Lemma 3.2.

$$\begin{aligned}
&P(\tau_{u(T_1)} \le T, X_T \le l \mid X_0 = 0) \\
&\quad = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_1 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} P(\tau_{u(T_1)} \le T, X_T \le l \mid X_{T_1} = x_1) dx_1 \\
&\qquad + \int_{u}^{\infty} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_2 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} P(\tau_{u(T_1)} \le T, X_T \le l \mid X_{T_1} = x_2) dx_2
\end{aligned}$$

When $X_{T_1} > u$, the event $\left\{\tau_{u(T_1)} \le T, X_T \le l\right\}$ is equivalent to $\{X_T \le l\}$. Thus

$$\begin{aligned}
&P(\tau_{u(T_1)} \le T, X_T \le l \mid X_0 = 0) \\
&\quad = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_1 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} P(\tau_{u(T_1)} \le T, X_T \le l \mid X_{T_1} = x_1) dx_1 \\
&\qquad + \int_{u}^{\infty} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_2 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} P(X_T \le l \mid X_{T_1} = x_2) dx_2 \\
&\quad = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_1 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} e^{\frac{2\mu}{\sigma}(u - x_1)} N\left(\frac{l - 2u + x_1 - \frac{\mu}{\sigma}(T - T_1)}{\sqrt{T - T_1}}\right) dx_1
\end{aligned}$$

$$+\int_u^\infty \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_2-\frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} N\left(\frac{l-x_2-\frac{\mu}{\sigma}(T-T_1)}{\sqrt{T-T_1}}\right) dx_2$$

$$= e^{\frac{2\mu u}{\sigma}} \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_1+\frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} N\left(\frac{l-2u+x_1-\frac{\mu}{\sigma}(T-T_1)}{\sqrt{T-T_1}}\right) dx_1$$

$$+\int_u^\infty \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_2-\frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} N\left(\frac{l-x_2-\frac{\mu}{\sigma}(T-T_1)}{\sqrt{T-T_1}}\right) dx_2$$

Applying Lemma 3.1 again to obtain

$$\begin{aligned} &P(\tau_{u(T_1)} \le T, X_T \le l \mid X_0 = 0) \\ &\quad = \exp\left(\frac{2\mu u}{\sigma}\right) N_2\left(\frac{u+\frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{l-2u-\frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right) \\ &\quad + N_2\left(\frac{-u+\frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{l-\frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right) \end{aligned} \tag{3.4}$$

□

## 3.1 Formulas for the option with barrier greater than strike price

We assume $X \le U$. The valuation formulas for the digitals in (3.1) are

$$\begin{aligned} &\mathcal{D}(S, t\,; A_7) \\ &= e^{-r(T-t)} \left(\frac{U}{S_t}\right)^{\frac{2\mu}{\sigma^2}} [G_1(X) - G_1(k)] + e^{-r(T-t)} \left(\frac{k}{S_t}\right)^{\frac{2\mu}{\sigma^2}} [G_2(X) - G_2(k)] \\ &\quad + e^{-r(T-t)} \left(\frac{k}{U}\right)^{\frac{2\mu}{\sigma^2}} [G_3(X) - G_3(k)] + e^{-r(T-t)} \Big[G_4(X) - G_4(k)\Big], \end{aligned}$$

$$\begin{aligned} &\mathcal{DS}(S, t\,; A_7) \\ &= S_t e^{-q(T-t)} \left(\frac{U}{S_t}\right)^{\frac{2\overline{\mu}}{\sigma^2}} \Big[\overline{G}_1(X) - \overline{G}_1(k)\Big] + S_t e^{-q(T-t)} \left(\frac{k}{S_t}\right)^{\frac{2\overline{\mu}}{\sigma^2}} \Big[\overline{G}_2(X) - \overline{G}_2(k)\Big] \\ &\quad + S_t e^{-q(T-t)} \left(\frac{k}{U}\right)^{\frac{2\overline{\mu}}{\sigma^2}} \Big[\overline{G}_3(X) - \overline{G}_3(k)\Big] + S_t e^{-q(T-t)} \Big[\overline{G}_4(X) - \overline{G}_4(k)\Big], \end{aligned}$$

$$\begin{aligned} &\mathcal{E}(S, t, k\,; A_8) \\ &= (X-k)\left[\left(\frac{U}{k}\right)^{\beta-b} \left(\frac{U}{S_t}\right)^{\beta+b} H_1(k) + \left(\frac{k}{S_t}\right)^{\beta+b} H_2(k) \right. \\ &\quad \left. + \left(\frac{S_t}{U}\right)^{\beta-b} \left(\frac{k}{U}\right)^{\beta+b} H_3(k) + \left(\frac{S_t}{k}\right)^{\beta-b} H_4(k)\right] \end{aligned}$$

where $G_i(X)$, $\overline{G}_i(X)$, and $H_i(k)(i = 1, \ldots, 4)$ are given in Theorems 3.1 and 3.3.

*Remark 3.1* When the barrier appears immediately after the option's initiation (i.e., $T_1$ converges to 0), it can be checked that the above formulas for $\mathcal{D}$, $\mathcal{DS}$ and $\mathcal{E}$ become the values of these digitals for American barrier option given in Sect. 2.

**Lemma 3.4** *For $l \le x \le u$, the probability that the process $X_t$ crosses $u$ after time $T_1$, and then does not fall below $l$ before expiration $T$, and its value at time $T$ is less than $x$ is*

$$\begin{aligned}
&P(\tau_{u(T_1)} < T,\ \tau_{ul(T_1)} > T,\ X_T \le x \mid X_0 = 0)\\
&\quad = \exp\left(\frac{2\mu u}{\sigma}\right)[F_1(x) - F_1(l)] + \exp\left(\frac{2\mu l}{\sigma}\right)[F_2(x) - F_2(l)]\\
&\qquad + \exp\left(\frac{2\mu}{\sigma}(l-u)\right)[F_3(x) - F_3(l)] + F_4(x) - F_4(l)
\end{aligned}$$

*where*

$$\begin{aligned}
F_1(x) &= N_2\left(\frac{u + \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{x - 2u - \frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right),\\
F_2(x) &= N_2\left(\frac{-u - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{2l - x + \frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right),\\
F_3(x) &= N_2\left(\frac{u - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{2l - 2u - x + \frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right),\\
F_4(x) &= N_2\left(\frac{-u + \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{x - \frac{\mu}{\sigma}T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right).
\end{aligned}$$

*Proof*

$$\begin{aligned}
&P(\tau_{u(T_1)} < T,\ \tau_{ul(T_1)} > T,\ X_T \le x \mid X_0 = 0)\\
&= P(\tau_{u(T_1)} < T,\ X_T \le x \mid X_0 = 0) - P(\tau_{u(T_1)} < T,\ \tau_{ul(T_1)} \le T,\ X_T \le x \mid X_0 = 0)\\
&= P(\tau_{u(T_1)} < T,\ X_T \le x \mid X_0 = 0) - P(\tau_{ul(T_1)} \le T,\ X_T \le x \mid X_0 = 0)\\
&= P(\tau_{u(T_1)} < T,\ X_T \le x \mid X_0 = 0) - P(\tau_{ul(T_1)} \le T \mid X_0 = 0)\\
&\quad + P(\tau_{ul(T_1)} \le T,\ X_T > x \mid X_0 = 0)
\end{aligned}$$

The first probability is obtained from (3.4) with $l = x$. The second and third probabilities are calculated by Lemmas 3.3 and 3.2. □

We now consider the digital options for American up-and-in put where the underlying asset price is monitored over the time period between $T_1$ and maturity $T$ under a constant exercise policy $k$. The values of these options are determined from the above Lemmas.

**Theorem 3.1** *For $X \leq U$, the values of a digital option and a digital share at time $t < T_1$ for the event $A_7 = \left\{\tau_{U(T_1)} < T,\ \tau_{Uk(T_1)} > T,\ S_T \leq X\right\}$ are*

$$
\begin{aligned}
&\mathcal{D}(S, t; A_7)\\
&= e^{-r(T-t)} \left(\frac{U}{S_t}\right)^{\frac{2\mu}{\sigma^2}} [G_1(X) - G_1(k)] + e^{-r(T-t)} \left(\frac{k}{S_t}\right)^{\frac{2\mu}{\sigma^2}} [G_2(X) - G_2(k)]\\
&\quad + e^{-r(T-t)} \left(\frac{k}{U}\right)^{\frac{2\mu}{\sigma^2}} [G_3(X) - G_3(k)] + e^{-r(T-t)}[G_4(X) - G_4(k)],\\
&\mathcal{DS}(S, t; A_7)\\
&= S_t e^{-q(T-t)} \left(\frac{U}{S_t}\right)^{\frac{2\overline{\mu}}{\sigma^2}} \left[\overline{G}_1(X) - \overline{G}_1(k)\right] + S_t e^{-q(T-t)} \left(\frac{k}{S_t}\right)^{\frac{2\overline{\mu}}{\sigma^2}} \left[\overline{G}_2(X) - \overline{G}_2(k)\right]\\
&\quad + S_t e^{-q(T-t)} \left(\frac{k}{U}\right)^{\frac{2\overline{\mu}}{\sigma^2}} \left[\overline{G}_3(X) - \overline{G}_3(k)\right] + S_t e^{-q(T-t)}\left[\overline{G}_4(X) - \overline{G}_4(k)\right]
\end{aligned}
$$

*where*

$$
\begin{aligned}
G_1(X) &= N_2\left(h_3\left(\frac{U}{S_t}\right), -h_1\left(\frac{U^2}{S_t X}\right); -\sqrt{\frac{T_1 - t}{T - t}}\right),\\
G_2(X) &= N_2\left(-h_3\left(\frac{U}{S_t}\right), h_1\left(\frac{k^2}{S_t X}\right); -\sqrt{\frac{T_1 - t}{T - t}}\right),\\
G_3(X) &= N_2\left(-h_3\left(\frac{S_t}{U}\right), h_1\left(\frac{S_t k^2}{U^2 X}\right); -\sqrt{\frac{T_1 - t}{T - t}}\right),\\
G_4(X) &= N_2\left(h_3\left(\frac{S_t}{U}\right), -h_1\left(\frac{S_t}{X}\right); -\sqrt{\frac{T_1 - t}{T - t}}\right),
\end{aligned}
$$

*and*

$$
h_1(z) = \frac{\ln z + \mu(T - t)}{\sigma\sqrt{T - t}}, \quad h_3(z) = \frac{\ln z + \mu(T_1 - t)}{\sigma\sqrt{T_1 - t}}.
$$

*$\overline{G}_i(X)$ is the same as $G_i(X)$ except $\overline{\mu} = r - q + \frac{\sigma^2}{2}$ in replacement of $\mu$ for $i = 1, 2, 3, 4$.*

*Proof* Apply Lemma 3.4 with letting $u = \frac{1}{\sigma}\ln\frac{U}{S_t}$, $l = \frac{1}{\sigma}\ln\frac{k}{S_t}$, and $x = \frac{1}{\sigma}\ln\frac{X}{S_t}$ to derive the risk-neutral probability of exercise at maturity. Then

$$\begin{aligned}
&P(\tau_{U(T_1)} < T,\ \tau_{Uk(T_1)} > T,\ S_T \le X \mid S_t)\\
&\quad = \left(\frac{U}{S_t}\right)^{\frac{2\mu}{\sigma^2}}[G_1(X) - G_1(k)] + \left(\frac{k}{S_t}\right)^{\frac{2\mu}{\sigma^2}}[G_2(X) - G_2(k)]\\
&\qquad + \left(\frac{k}{U}\right)^{\frac{2\mu}{\sigma^2}}[G_3(X) - G_3(k)] + G_4(X) - G_4(k)
\end{aligned}$$

where $G_i(X)$ for $i = 1, 2, 3, 4$ are defined as above. Then the value of digital option $\mathcal{D}(S, t; A_7)$ at time $t$

$$\mathcal{D}(S, t; A_7) = e^{-r(T-t)}P(\tau_{U(T_1)} < T,\ \tau_{Uk(T_1)} > T,\ S_T \le X \mid S_t)$$

is obtained as desired. The digital share $\mathcal{DS}(S, t; A_7)$ can be valued by changing $\mu$ to $\overline{\mu} = r - q + \frac{\sigma^2}{2}$ and replacing the discount factor $e^{-r(T-t)}$ by $S_t e^{-q(T-t)}$ (See for example Ingersoll 2000). □

**Theorem 3.2** *The value of a digital option and a digital share at time t for the event* $A_8 = \{\tau_{Uk(T_1)} < T\}$ *are*

$$\begin{aligned}
&\mathcal{D}(S, t; A_8)\\
&\quad = e^{-r(T-t)}\left[\left(\frac{U}{S_t}\right)^{\frac{2\mu}{\sigma^2}} G_1(k) + \left(\frac{k}{S_t}\right)^{\frac{2\mu}{\sigma^2}} G_2(k) + \left(\frac{k}{U}\right)^{\frac{2\mu}{\sigma^2}} G_3(k) + G_4(k)\right],
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\mathcal{DS}(S, t; A_8)\\
&\quad = S_t e^{-q(T-t)}\left[\left(\frac{U}{S_t}\right)^{\frac{2\overline{\mu}}{\sigma^2}} \overline{G}_1(k) + \left(\frac{k}{S_t}\right)^{\frac{2\overline{\mu}}{\sigma^2}} \overline{G}_2(k) + \left(\frac{k}{U}\right)^{\frac{2\overline{\mu}}{\sigma^2}} \overline{G}_3(k) + \overline{G}_4(k)\right]
\end{aligned}$$

*Proof* Apply Lemma 3.3 with $u = \frac{1}{\sigma}\ln\frac{U}{S_t}$, $l = \frac{1}{\sigma}\ln\frac{k}{S_t}$, and $x = \frac{1}{\sigma}\ln\frac{X}{S_t}$ to derive the risk-neutral probability of early exercise. Then

$$P(\tau_{Uk(T_1)} \le T \mid S_t) = \left(\frac{U}{S_t}\right)^{\frac{2\mu}{\sigma^2}} G_1(k) + \left(\frac{k}{S_t}\right)^{\frac{2\mu}{\sigma^2}} G_2(k) + \left(\frac{k}{U}\right)^{\frac{2\mu}{\sigma^2}} G_3(k) + G_4(k)$$

Thus, the value of digital option at time $t$

$$\mathcal{D}(S, t; A_8) = e^{-r(T-t)}P(\tau_{Uk(T_1)} \le T \mid S_t)$$

is obtained. The digital share, $\mathcal{DS}(S, t; A_8)$ can be valued as in Theorem 3.1. □

Under a constant exercise policy, the up-and-in put option will be exercised early prior to maturity for $X-k$ if the stock price hits the up-barrier $U$ after $T_1$, and then falls to $k$ after $\tau_{U(T_1)}$ before maturity. Now we consider the value of a first-touch digital for time $\tau_{Uk(T_1)}$. We examine the case when there is no dividend on the stock first.

**Lemma 3.5** *If the stock does not pay dividends, the value of a first-touch digital for the event* $A_8 = \{\tau_{Uk(T_1)} < T\}$ *is*

$$\mathcal{E}(S,t,k;A_8) = \frac{X-k}{k} S_t \left[ \left(\frac{U}{S_t}\right)^{\frac{2r}{\sigma^2}+1} \widetilde{G}_1(k) + \left(\frac{k}{S_t}\right)^{\frac{2r}{\sigma^2}+1} \widetilde{G}_2(k) + \left(\frac{k}{U}\right)^{\frac{2r}{\sigma^2}+1} \widetilde{G}_3(k) + \widetilde{G}_4(k) \right]$$

*where* $\widetilde{G}_i(k)$ *is the same as* $G_i(k)$ *except* $\widetilde{\mu} = r + \frac{1}{2}\sigma^2$ *in replacement of* $\mu$ *for* $i = 1, 2, 3, 4$.

*Proof* The first-touch digital pays $X-k$ at time $\tau_{Uk(T_1)}$. This money can be used to purchase $\frac{X-k}{k}$ shares of the stock at that time. Since the shares do not pay dividends, it is worth $\frac{X-k}{k} S_T$ at maturity, i.e.,

$$\mathcal{E}(S,t,k;A_8) = \frac{X-k}{k} \mathcal{DS}(S,t;A_8)$$

where $\mathcal{DS}(S,t;A_8)$ is the value when $q=0$ in (3.5). □

**Theorem 3.3** *The value of the first-touch digital for the event* $A_8$ *is*

$$\begin{aligned}\mathcal{E}(S,t,k;A_8) = (X-k) \Bigg[ &\left(\frac{U}{k}\right)^{\beta-b} \left(\frac{U}{S_t}\right)^{\beta+b} H_1(k) + \left(\frac{k}{S_t}\right)^{\beta+b} H_2(k) \\ &+ \left(\frac{S_t}{U}\right)^{\beta-b} \left(\frac{k}{U}\right)^{\beta+b} H_3(k) + \left(\frac{S_t}{k}\right)^{\beta-b} H_4(k) \Bigg]\end{aligned}$$

*where*

$$H_1(k) = N_2\left( g_2\left(\frac{U}{S_t}\right), -g_1\left(\frac{U^2}{S_t k}\right); -\sqrt{\frac{T_1-t}{T-t}} \right),$$

$$H_2(k) = N_2\left( -g_2\left(\frac{U}{S_t}\right), g_1\left(\frac{k}{S_t}\right); -\sqrt{\frac{T_1-t}{T-t}} \right),$$

$$H_3(k) = N_2\left( -g_2\left(\frac{S_t}{U}\right), g_1\left(\frac{S_t k}{U^2}\right); -\sqrt{\frac{T_1-t}{T-t}} \right),$$

$$H_4(k) = N_2\left( g_2\left(\frac{S_t}{U}\right), -g_1\left(\frac{S_t}{k}\right); -\sqrt{\frac{T_1-t}{T-t}} \right),$$

*and*

$$g_1(z) = \frac{\ln z + \beta\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \quad g_2(z) = \frac{\ln z + \beta\sigma^2(T_1-t)}{\sigma\sqrt{T_1-t}}.$$

*Proof* When the stock price pays dividends, the asset price follows the continuous diffusion process $dS_t = (r-q)S_t dt + \sigma S_t dW$. To eliminate the dividend term in the process, we set

$$V_t = S_t^{\beta-b}$$

where

$$b = \frac{\mu}{\sigma^2} \text{ and } \beta = \sqrt{b^2 + \frac{2r}{\sigma^2}}.$$

Then, by Ito's lemma,

$$dV_t = rV_t dt + (\beta-b)\sigma V_t dW_t. \tag{3.6}$$

We may apply Lemma 3.5 to the process $V_t$ since (3.6) does not contain the dividend term. The barriers for $V_t$ corresponding to $U$ and $k$ are $U^{\beta-b}$ and $k^{\beta-b}$. Furthermore, the volatility $\sigma$ is replaced by $(\beta-b)\sigma$. Then the value of the first-touch digital for the event $A_8$ is

$$\begin{aligned}\mathcal{E}(V,t,k^{\beta-b};A_8) = \frac{X-k}{k^{\beta-b}}V_t\Bigg[&\left(\frac{U^{\beta-b}}{V_t}\right)^{\frac{2r}{(\beta-b)^2\sigma^2}+1}H_1(k) + \left(\frac{k^{\beta-b}}{V_t}\right)^{\frac{2r}{(\beta-b)^2\sigma^2}+1}H_2(k)\\ &+\left(\frac{k^{\beta-b}}{U^{\beta-b}}\right)^{\frac{2r}{(\beta-b)^2\sigma^2}+1}H_3(k) + H_4(k)\Bigg]\end{aligned}$$

where $\widehat{\mu} = r + \frac{1}{2}(\beta-b)^2\sigma^2$,

$$H_1(k) = N_2\left(\frac{\ln\left(\frac{U^{\beta-b}}{V_t}\right) + \widehat{\mu}(T_1-t)}{(\beta-b)\sigma\sqrt{T_1-t}}, \frac{\ln\left(\frac{k^{\beta-b}V_t}{U^{2(\beta-b)}}\right) - \widehat{\mu}(T-t)}{(\beta-b)\sigma\sqrt{T-t}}; -\sqrt{\frac{T_1-t}{T-t}}\right),$$

$$H_2(k) = N_2\left(\frac{-\ln\left(\frac{U^{\beta-b}}{V_t}\right) - \widehat{\mu}(T_1-t)}{(\beta-b)\sigma\sqrt{T_1-t}}, \frac{\ln\left(\frac{k^{\beta-b}}{V_t}\right) + \widehat{\mu}(T-t)}{(\beta-b)\sigma\sqrt{T-t}}; -\sqrt{\frac{T_1-t}{T-t}}\right),$$

$$H_3(k) = N_2\left(\frac{\ln\left(\frac{U^{\beta-b}}{V_t}\right) - \widehat{\mu}(T_1-t)}{(\beta-b)\sigma\sqrt{T_1-t}}, \frac{\ln\left(\frac{k^{\beta-b}V_t}{U^{2(\beta-b)}}\right) + \widehat{\mu}(T-t)}{(\beta-b)\sigma\sqrt{T-t}}; -\sqrt{\frac{T_1-t}{T-t}}\right),$$

$$H_4(k) = N_2\left(\frac{-\ln\left(\frac{U^{\beta-b}}{V_t}\right) + \widehat{\mu}(T_1 - t)}{(\beta - b)\sigma\sqrt{T_1 - t}}, \frac{\ln\left(\frac{k^{\beta-b}}{V_t}\right) - \widehat{\mu}(T - t)}{(\beta - b)\sigma\sqrt{T - t}}; -\sqrt{\frac{T_1 - t}{T - t}}\right).$$

Thus,

$$\begin{aligned}
&\mathcal{E}(S, t, k; A_8)\\
&= (X - k)\left(\frac{S_t}{k}\right)^{\beta-b}\left[\left(\frac{U}{S_t}\right)^{(\beta-b)\left(\frac{2r}{(\beta-b)^2\sigma^2}+1\right)} H_1(k) + \left(\frac{k}{S_t}\right)^{(\beta-b)\left(\frac{2r}{(\beta-b)^2\sigma^2}+1\right)} H_2(k)\right.\\
&\quad\left. + \left(\frac{k}{U}\right)^{(\beta-b)\left(\frac{2r}{(\beta-b)^2\sigma^2}+1\right)} H_3(k) + H_4(k)\right]\\
&= (X - k)\left[\left(\frac{U}{k}\right)^{\beta-b}\left(\frac{U}{S_t}\right)^{\beta+b} H_1(k) + \left(\frac{k}{S_t}\right)^{\beta+b} H_2(k)\right.\\
&\quad\left. + \left(\frac{S_t}{U}\right)^{\beta-b}\left(\frac{k}{U}\right)^{\beta+b} H_3(k) + \left(\frac{S_t}{k}\right)^{\beta-b} H_4(k)\right]
\end{aligned}$$

where

$$\begin{aligned}
H_1(k) &= N_2\left(g_2\left(\frac{U}{S_t}\right), -g_1\left(\frac{U^2}{S_t k}\right); -\sqrt{\frac{T_1 - t}{T - t}}\right),\\
H_2(k) &= N_2\left(-g_2\left(\frac{U}{S_t}\right), g_1\left(\frac{k}{S_t}\right); -\sqrt{\frac{T_1 - t}{T - t}}\right),\\
H_3(k) &= N_2\left(-g_2\left(\frac{S_t}{U}\right), g_1\left(\frac{S_t k}{U^2}\right); -\sqrt{\frac{T_1 - t}{T - t}}\right),\\
H_4(k) &= N_2\left(g_2\left(\frac{S_t}{U}\right), -g_1\left(\frac{S_t}{k}\right); -\sqrt{\frac{T_1 - t}{T - t}}\right),
\end{aligned}$$

and

$$g_1(z) = \frac{\ln z + \beta\sigma^2(T - t)}{\sigma\sqrt{T - t}}, \quad g_2(z) = \frac{\ln z + \beta\sigma^2(T_1 - t)}{\sigma\sqrt{T_1 - t}}.$$

□

The following graph, Fig. 1, illustrates the American up-and-in put prices using the approximation (3.1) with different values of initial spot $S_0$ and barrier's starting time $T_1$. Also, Fig. 2 shows the option prices with different values of up-barrier $U$ and $T_1$.

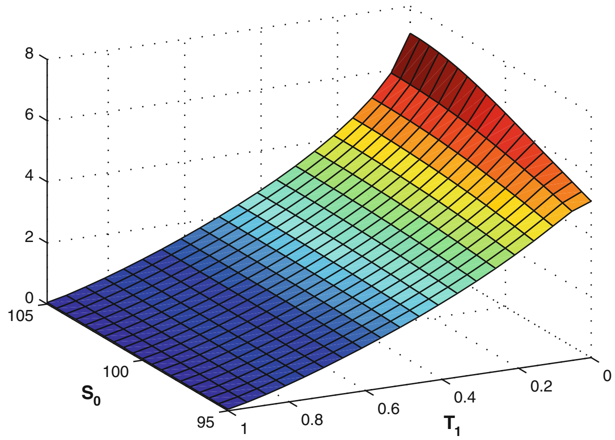


**Fig. 1** $PUIP_{\text{const}}$ result, varying $S_0$ and $T_1$ when $U \geq X$ (option parameters: $U = 105,\ X = 100,\ r = 0.05,\ \sigma = 0.3$, and $T = 1$)

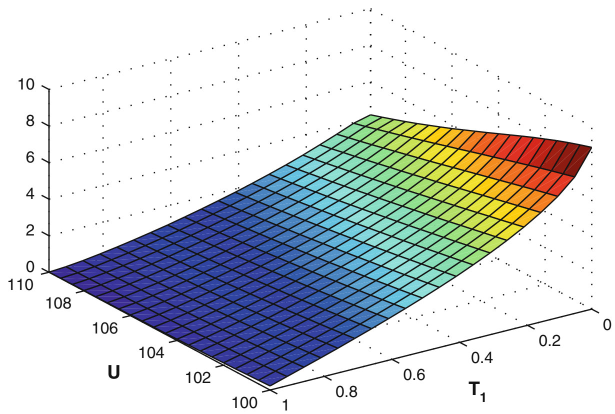


**Fig. 2** $PUIP_{\text{const}}$ result, varying $U$ and $T_1$ when $U \geq X$ (option parameters: $S_0 = 100,\ X = 100,\ r = 0.05,\ \sigma = 0.3$, and $T = 1$)

## 3.2 Formulas for the option with barrier less than strike price

We assume $U < X$. The valuation formulas for the digitals in (3.1) are

$$\begin{aligned}\mathcal{D}(S, t; A_7) =& e^{-r(T-t)}\left(\frac{U}{S_t}\right)^{\frac{2\mu}{\sigma^2}}\left[G_1(U) - G_1(k) + G_5(X) - G_5(U)\right]\\&+ e^{-r(T-t)}\left(\frac{k}{S_t}\right)^{\frac{2\mu}{\sigma^2}}\left[G_2(X) - G_2(k)\right] + e^{-r(T-t)}\left(\frac{k}{U}\right)^{\frac{2\mu}{\sigma^2}}\\&\times\left[G_3(X) - G_3(k)\right] + e^{-r(T-t)}\Big[G_4(U) - G_4(k) + G_6(X) - G_6(U)\Big],\end{aligned}$$

$$\begin{aligned}
&\mathcal{DS}(S,t;A_7)\\
&= S_t e^{-q(T-t)}\left(\frac{U}{S_t}\right)^{\frac{2\overline{\mu}}{\sigma^2}}\left[\overline{G}_1(U)-\overline{G}_1(k)+\overline{G}_5(X)-\overline{G}_5(U)\right]\\
&\quad + S_t e^{-q(T-t)}\left(\frac{k}{S_t}\right)^{\frac{2\overline{\mu}}{\sigma^2}}\left[\overline{G}_2(X)-\overline{G}_2(k)\right]+S_t e^{-q(T-t)}\left(\frac{k}{U}\right)^{\frac{2\overline{\mu}}{\sigma^2}}\left[\overline{G}_3(X)-\overline{G}_3(k)\right]\\
&\quad + S_t e^{-q(T-t)}[\overline{G}_4(U)-\overline{G}_4(k)+\overline{G}_6(X)-\overline{G}_6(U)],
\end{aligned}$$

$$\begin{aligned}
&\mathcal{E}(S,t,k\,;A_8)\\
&=(X-k)\left[\left(\frac{U}{k}\right)^{\beta-b}\left(\frac{U}{S_t}\right)^{\beta+b}H_1(k)+\left(\frac{k}{S_t}\right)^{\beta+b}H_2(k)\right.\\
&\quad\left.+\left(\frac{S_t}{U}\right)^{\beta-b}\left(\frac{k}{U}\right)^{\beta+b}H_3(k)+\left(\frac{S_t}{k}\right)^{\beta-b}H_4(k)\right]
\end{aligned}$$

where $G_i(X)$, $\overline{G}_i(X)(i=1,\ldots,6)$, and $H_i(k)(i=1,\ldots,4)$ are given in Theorems 3.1, 3.3 and 3.4.

**Lemma 3.6** *For $x>u$, the probability that the process $X_t$ crosses $u$ after $T_1$, and then does not fall below $l$ before time $T$, and its value at time $T$ is less than $x$ is*

$$\begin{aligned}
&P(\tau_{u(T_1)}<T,\ \tau_{ul(T_1)}>T,\ X_T\le x\mid X_0=0)\\
&=\exp\left(\frac{2\mu u}{\sigma}\right)[F_1(u)-F_1(l)+F_5(x)-F_5(u)]+\exp\left(\frac{2\mu l}{\sigma}\right)[F_2(x)-F_2(l)]\\
&\quad+\exp\left(\frac{2\mu}{\sigma}(l-u)\right)[F_3(x)-F_3(l)]+F_4(u)-F_4(l)+F_6(x)-F_6(u)
\end{aligned}$$

*where*

$$F_5(x)=N_2\left(\frac{-u-\frac{\mu}{\sigma}T_1}{\sqrt{T_1}},\frac{x-2u-\frac{\mu}{\sigma}T}{\sqrt{T}};\sqrt{\frac{T_1}{T}}\right),$$
$$F_6(x)=N_2\left(\frac{u-\frac{\mu}{\sigma}T_1}{\sqrt{T_1}},\frac{x-\frac{\mu}{\sigma}T}{\sqrt{T}};\sqrt{\frac{T_1}{T}}\right).$$

*Proof*

$$\begin{aligned}
&P(\tau_{u(T_1)}<T,\ \tau_{ul(T_1)}>T,\ X_T\le x\mid X_0=0)\\
&=P(\tau_{u(T_1)}<T,\ X_T\le x\mid X_0=0)-P(\tau_{ul(T_1)}\le T\mid X_0=0)\\
&\quad+P(\tau_{ul(T_1)}\le T,\ X_T>x\mid X_0=0)\\
&=P(\tau_{u(T_1)}<T,\ X_T\le u\mid X_0=0)+P(\tau_{u(T_1)}<T,\ u<X_T\le x\mid X_0=0)\\
&\quad-P(\tau_{ul(T_1)}\le T\mid X_0=0)+P(\tau_{ul(T_1)}\le T,\ X_T>x\mid X_0=0).
\end{aligned}$$

The third and fourth probabilities above are calculated by Lemmas 3.3 and 3.2. The first probability comes from (3.4) with a replacement of $l$ by $u$. Thus we only need to prove the second probability.

$$
\begin{aligned}
&P(\tau_{u(T_1)} < T,\ u < X_T \le x \mid X_0 = 0)\\
&= \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_1 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} P(\tau_{u(T_1)} < T,\ u < X_T \le x \mid X_{T_1} = x_1) dx_1\\
&\quad + \int_{u}^{\infty} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_2 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} P(\tau_{u(T_1)} < T,\ u < X_T \le x \mid X_{T_1} = x_2) dx_2\\
&= \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_1 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} P(u < X_T \le x \mid X_{T_1} = x_1) dx_1\\
&\quad + \int_{u}^{\infty} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_2 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} P(\tau_{u(T_1)} < T,\ u < X_T \le x \mid X_{T_1} = x_2) dx_2\\
&= \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_1 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} \left[ N\left(\frac{x - x_1 - \frac{\mu}{\sigma}(T - T_1)}{\sqrt{T - T_1}}\right)\right.\\
&\quad \left. - N\left(\frac{u - x_1 - \frac{\mu}{\sigma}(T - T_1)}{\sqrt{T - T_1}}\right)\right] dx_1 + \int_{u}^{\infty} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2}\left(\frac{x_2 - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}\right)^2} e^{\frac{2\mu}{\sigma}(u - x_2)}\\
&\quad \times \left[ N\left(\frac{x - 2u + x_2 - \frac{\mu}{\sigma}(T - T_1)}{\sqrt{T - T_1}}\right) - N\left(\frac{-u + x_2 - \frac{\mu}{\sigma}(T - T_1)}{\sqrt{T - T_1}}\right)\right] dx_2\\
&= N_2\left(\frac{u - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{x - \frac{\mu}{\sigma}T}{\sqrt{T}}; \sqrt{\frac{T_1}{T}}\right) - N_2\left(\frac{u - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{u - \frac{\mu}{\sigma}T}{\sqrt{T}}; \sqrt{\frac{T_1}{T}}\right)\\
&\quad + e^{\frac{2\mu u}{\sigma}} \left[ N_2\left(\frac{-u - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{x - 2u - \frac{\mu}{\sigma}T}{\sqrt{T}}; \sqrt{\frac{T_1}{T}}\right)\right.\\
&\quad \left. - N_2\left(\frac{-u - \frac{\mu}{\sigma}T_1}{\sqrt{T_1}}, \frac{-u - \frac{\mu}{\sigma}T}{\sqrt{T}}; \sqrt{\frac{T_1}{T}}\right)\right]
\end{aligned}
$$

□

**Theorem 3.4** *For $X > U$, the values of a digital option and a digital share at time $t < T_1$ for the event $A_7 = \left\{\tau_{U(T_1)} < T,\ \tau_{Uk(T_1)} > T,\ S_T \le X\right\}$ are*

$$
\mathcal{D}(S, t; A_7) = e^{-r(T-t)} \left(\frac{U}{S_t}\right)^{\frac{2\mu}{\sigma^2}} [G_1(U) - G_1(k) + G_5(X) - G_5(U)]
$$

$$+ e^{-r(T-t)} \left(\frac{k}{S_t}\right)^{\frac{2\mu}{\sigma^2}} [G_2(X) - G_2(k)] + e^{-r(T-t)} \left(\frac{k}{U}\right)^{\frac{2\mu}{\sigma^2}}$$
$$\times [G_3(X) - G_3(k)] + e^{-r(T-t)} \Big[G_4(U) - G_4(k) + G_6(X) - G_6(U)\Big],$$

$$\mathcal{DS}(S, t; A_7)$$
$$= S_t e^{-q(T-t)} \left(\frac{U}{S_t}\right)^{\frac{2\overline{\mu}}{\sigma^2}} \left[\overline{G}_1(U) - \overline{G}_1(k) + \overline{G}_5(X) - \overline{G}_5(U)\right]$$
$$+ S_t e^{-q(T-t)} \left(\frac{k}{S_t}\right)^{\frac{2\overline{\mu}}{\sigma^2}} \left[\overline{G}_2(X) - \overline{G}_2(k)\right] + S_t e^{-q(T-t)} \left(\frac{k}{U}\right)^{\frac{2\overline{\mu}}{\sigma^2}} \left[\overline{G}_3(X) - \overline{G}_3(k)\right]$$
$$+ S_t e^{-q(T-t)} \Big[\overline{G}_4(U) - \overline{G}_4(k) + \overline{G}_6(X) - \overline{G}_6(U)\Big]$$

*where*

$$G_5(X) = N_2\left(-h_3\left(\frac{U}{S_t}\right), -h_1\left(\frac{U^2}{S_t X}\right); \sqrt{\frac{T_1 - t}{T - t}}\right),$$
$$G_6(X) = N_2\left(h_3\left(\frac{S_t}{U}\right), -h_1\left(\frac{S_t}{X}\right); \sqrt{\frac{T_1 - t}{T - t}}\right).$$

*$\overline{G}_i(X)$ is the same as $G_i(X)$ except $\overline{\mu} = r - q + \frac{\sigma^2}{2}$ in replacement of $\mu$ for $i = 1, \ldots, 6$.*

*Proof* Apply Lemma 3.6 with having $u = \frac{1}{\sigma} \ln \frac{U}{S_t}$, $l = \frac{1}{\sigma} \ln \frac{k}{S_t}$, and $x = \frac{1}{\sigma} \ln \frac{X}{S_t}$. Then we obtain the result similarly as in the proof of Theorem 3.1. □

In the following, Fig. 3 illustrates the American up-and-in put prices using (3.1) with different values of initial spot $S_0$ and $T_1$ when $U < X$. Also, Fig. 4 shows the option prices with varying up-barrier $U$ and $T_1$.

We next present the values of American partial up-and-in put option by our formulae and compare them with those by Monte Carlo method with an Antithetic Variate (See for example Glasserman 2003) and by the Trinomial lattice model using the adaptive mesh model (AMM).[1] Table 1 shows the values of American partial up-and-in put option whose monitoring period begins at predetermined time $T_1$ with varying initial price $S_0$ and strike price $X$. The parameter values that we used are $U = 105$, $\sigma = 0.3$, $T = 0.5$, $T_1 = 0.1$, $r = 0.05$ and $q = 0$. The values of $S_0$ vary from 96 to 104 and the values of $X$ from 95 to 105. The values $PUIP_{\text{const}}$ in Table 1 are calculated by the formulae in Sect. 3.1. Table 2 shows the values of American partial up-and-in put option with different levels of upper barrier $U$ and time $T_1$. The parameter values in this computation are $S_0 = 100$, $X = 105$, $\sigma = 0.3$, $T = 0.5$, $r = 0.05$

[1] The adaptive mesh method (Figlewski and Gao 1999) sharply reduces nonlinearity error by grafting one or more small sections of fine high-resolution lattice onto a tree with coarser time and price steps.

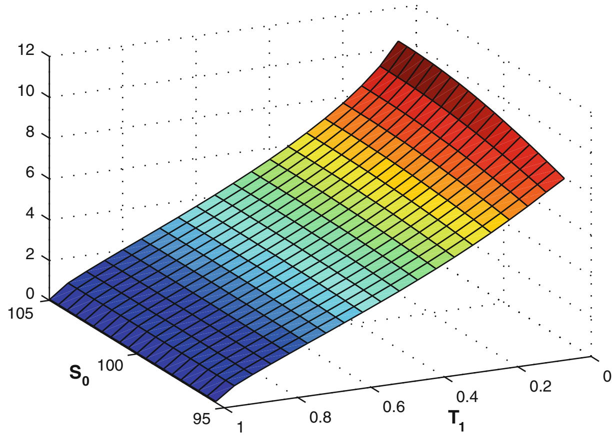


**Fig. 3** $PUIP_{\text{const}}$ result, varying $S_0$ and $T_1$ when $U < X$ (option parameters: $U = 105$, $X = 110$, $r = 0.05$, $\sigma = 0.3$, and $T = 1$)

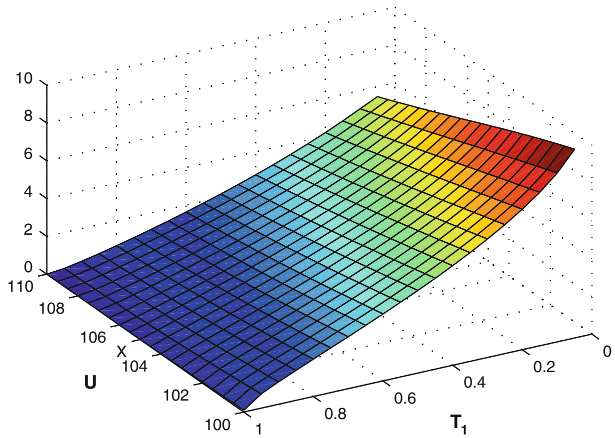


**Fig. 4** $PUIP_{\text{const}}$ result, varying $U$ and $T_1$ when $U < X$ (option parameters: $S_0 = 100$, $X = 110$, $r = 0.05$, $\sigma = 0.3$, and $T = 1$)

and $q = 0$. The values of $U$ vary from 102 to 108 and the values of $T_1$ from 0.1 to 0.4. The values $PUIP_{\text{const}}$ in Table 2 are computed by using the formulae in Sects. 3.1 and 3.2.

$V(N)$ is an option value of $PUIP_{\text{const}}$ using barrier options with constant policy barriers in (3.1). $N$ is the element number of constant policy set $\mathcal{K}_c$ to seek the best policy where policies are evenly spaced from 0 to $X$. Since the American put option comes into action only if the up-barrier is hit after $T_1$, the option price $PUIP_{\text{const}}$ decreases as the initial stock price gets farther apart from the up-barrier $U$. We notice that as the number $N$ of constant exercise policies increases, the option value $V(N)$ converges to a constant very quickly, as shown in Fig. 5.

**Table 1** Comparison of American partial barrier put option values $PUIP$ with varying $S_0$ and strike price $X$

| $S_0$ | X | V(10) | V(30) | V(50) | V(100) | V(500) | MC | AMM5 | $k^*$ |
|---|---|---|---|---|---|---|---|---|---|
| 96 | 95 | 1.3703 | 1.3736 | 1.3732 | 1.3736 | 1.3736 | 1.3878 | 1.3678 | 78.9925 |
| | 97.5 | 1.7900 | 1.7937 | 1.7934 | 1.7939 | 1.7939 | 1.8000 | 1.7871 | 80.7495 |
| | 100 | 2.2954 | 2.2989 | 2.2995 | 2.2995 | 2.2997 | 2.2973 | 2.2919 | 82.4900 |
| | 102.5 | 2.8939 | 2.8963 | 2.8983 | 2.8983 | 2.8984 | 2.8637 | 2.8897 | 84.2140 |
| | 105 | 3.5918 | 3.5918 | 3.5961 | 3.5961 | 3.5961 | 3.5369 | 3.5869 | 85.9215 |
| 98 | 95 | 1.4967 | 1.5003 | 1.4997 | 1.5003 | 1.5003 | 1.5123 | 1.4971 | 78.9450 |
| | 97.5 | 1.9493 | 1.9532 | 1.9530 | 1.9534 | 1.9535 | 1.9332 | 1.9498 | 80.7007 |
| | 100 | 2.4923 | 2.4959 | 2.4967 | 2.4967 | 2.4969 | 2.4654 | 2.4933 | 82.4300 |
| | 102.5 | 3.1330 | 3.1351 | 3.1377 | 3.1377 | 3.1377 | 3.1524 | 3.1340 | 84.1525 |
| | 105 | 3.8774 | 3.8774 | 3.8817 | 3.8817 | 3.8818 | 3.8249 | 3.8792 | 85.8585 |
| 100 | 95 | 1.5969 | 1.6007 | 1.6001 | 1.6007 | 1.6007 | 1.6042 | 1.6018 | 78.9070 |
| | 97.5 | 2.0753 | 2.0793 | 2.0792 | 2.0796 | 2.0797 | 2.0918 | 2.0815 | 80.6617 |
| | 100 | 2.6477 | 2.6513 | 2.6523 | 2.6523 | 2.6525 | 2.6504 | 2.6553 | 82.3900 |
| | 102.5 | 3.3213 | 3.3232 | 3.3261 | 3.3261 | 3.3261 | 3.3026 | 3.3304 | 84.1115 |
| | 105 | 4.1018 | 4.1018 | 4.1062 | 4.1062 | 4.1064 | 4.0656 | 4.1125 | 85.8060 |
| 102 | 95 | 1.6653 | 1.6693 | 1.6687 | 1.6693 | 1.6693 | 1.6754 | 1.6714 | 78.8880 |
| | 97.5 | 2.1612 | 2.1653 | 2.1653 | 2.1657 | 2.1658 | 2.1535 | 2.1687 | 80.6325 |
| | 100 | 2.7535 | 2.7571 | 2.7583 | 2.7583 | 2.7584 | 2.7357 | 2.7622 | 82.3700 |
| | 102.5 | 3.4493 | 3.4510 | 3.4542 | 3.4542 | 3.4542 | 3.4204 | 3.4587 | 84.0807 |
| | 105 | 4.2542 | 4.2542 | 4.2586 | 4.2586 | 4.2588 | 4.2626 | 4.2637 | 85.7745 |
| 104 | 95 | 1.6989 | 1.7029 | 1.7024 | 1.7030 | 1.7030 | 1.7109 | 1.7006 | 78.8785 |
| | 97.5 | 2.2033 | 2.2075 | 2.2075 | 2.2078 | 2.2079 | 2.2134 | 2.2153 | 80.6227 |
| | 100 | 2.8053 | 2.8089 | 2.8101 | 2.8101 | 2.8102 | 2.8181 | 2.8265 | 82.3500 |
| | 102.5 | 3.5118 | 3.5134 | 3.5168 | 3.5168 | 3.5168 | 3.4879 | 3.5249 | 84.0602 |
| | 105 | 4.3284 | 4.3284 | 4.3328 | 4.3328 | 4.3330 | 4.3276 | 4.3196 | 85.7640 |

Option parameters: $U = 105$, $T_1 = 0.1$, $T = 0.5$, $\sigma = 0.3$, $r = 0.05$, $q = 0$. $V(N)$ is an option value of $PUIP_{\text{const}}$ where $N$ is the number of constant policy barriers. MC is a result of simulation using the Antithetic Variates, a Variance Reduction Method of Monte Carlo simulation. AMM5 is a result of Trinomial lattice method by the AMM with level 5. $k^*$ is the optimal policy barrier for $V(10000)$

MC is a result of simulation using the Antithetic Variates, a Variance Reduction Method of Monte Carlo simulation. For the American partial barrier option using policy barriers, Monte Carlo method requires much larger amount of computer time because a large number of sample paths and policy barriers, and a large enough monitoring frequency must be needed. For the Monte Carlo approximation in Tables 1 and 2, the computer time is more than 10,000 times as long as for our formulae method to obtain the similar results under the same policy numbers. For the MC results in Tables 1 and 2, a monitoring frequency is 1,000, the number of sample paths is 5,000, and the number of policy barriers (evenly spaced from 0 to $X$) is 100.

AMM5 is an outcome of Trinomial lattice model by the adaptive mesh model presented in Figlewski and Gao (1999). This is the approach for constructing a lattice-

**Table 2** Comparison of American partial barrier put option values $PUIP$ with varying $U$ and $T_1$

| $U$ | $T_1$ | V(10) | V(30) | V(50) | V(100) | V(500) | MC | AMM5 | $k^*$ |
|---|---|---|---|---|---|---|---|---|---|
| 102 | 0.1 | 5.3817 | 5.3817 | 5.3837 | 5.3851 | 5.3852 | 5.3874 | 5.3743 | 85.3230 |
| | 0.2 | 3.8061 | 3.8169 | 3.8152 | 3.8170 | 3.8170 | 3.8102 | 3.8358 | 87.2340 |
| | 0.3 | 2.5479 | 2.5579 | 2.5603 | 2.5610 | 2.5611 | 2.5616 | 2.5567 | 89.5545 |
| | 0.4 | 1.4306 | 1.4319 | 1.4333 | 1.4333 | 1.4334 | 1.4301 | 1.4026 | 92.8830 |
| 104 | 0.1 | 4.5022 | 4.5022 | 4.5061 | 4.5061 | 4.5065 | 4.5101 | 4.4840 | 85.6485 |
| | 0.2 | 3.1057 | 3.1157 | 3.1150 | 3.1155 | 3.1157 | 3.1314 | 3.0931 | 87.5490 |
| | 0.3 | 1.9848 | 1.9938 | 1.9950 | 1.9950 | 1.9952 | 1.9782 | 1.9721 | 89.8800 |
| | 0.4 | 1.0050 | 1.0050 | 1.0058 | 1.0061 | 1.0061 | 1.0111 | 0.9847 | 93.2400 |
| 106 | 0.1 | 3.7235 | 3.7237 | 3.7282 | 3.7282 | 3.7282 | 3.7152 | 3.7320 | 85.9635 |
| | 0.2 | 2.5264 | 2.5350 | 2.5349 | 2.5349 | 2.5351 | 2.5285 | 2.5354 | 87.8430 |
| | 0.3 | 1.5514 | 1.5588 | 1.5593 | 1.5593 | 1.5594 | 1.5496 | 1.5569 | 90.1950 |
| | 0.4 | 0.7130 | 0.7130 | 0.7131 | 0.7134 | 0.7134 | 0.7114 | 0.7099 | 93.5760 |
| 108 | 0.1 | 3.0181 | 3.0207 | 3.0228 | 3.0228 | 3.0228 | 3.0042 | 3.0068 | 86.2890 |
| | 0.2 | 2.0242 | 2.0312 | 2.0315 | 2.0315 | 2.0315 | 2.0362 | 2.0185 | 88.1475 |
| | 0.3 | 1.1960 | 1.2018 | 1.2019 | 1.2019 | 1.2019 | 1.1914 | 1.1909 | 90.4890 |
| | 0.4 | 0.4971 | 0.4971 | 0.4971 | 0.4971 | 0.4972 | 0.4936 | 0.4894 | 93.8910 |

Option parameters: $S_0 = 100,\ X = 105,\ \sigma = 0.3,\ T = 0.5,\ r = 0.05,\ q = 0$. $V(N)$ is an option value of $PUIP_{\text{const}}$ where $N$ is the number of constant policy barriers. MC is a result of simulation using the Antithetic variates, a Variance Reduction Method of Monte Carlo simulation. AMM5 is a result of trinomial lattice method by the AMM with level 5. $k^*$ is the optimal policy barrier for $V(10000)$

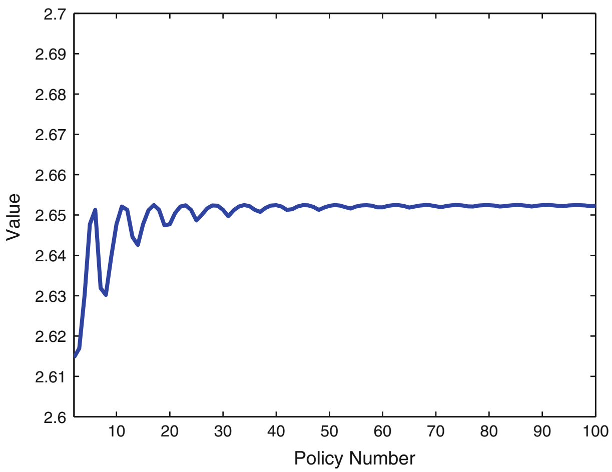


**Fig. 5** $PUIP_{\text{const}}$ result, varying policy barrier number $N$ (option parameters: $S_0 = 100,\ X = 100,\ U = 105,\ r = 0.05,\ \sigma = 0.3,\ T_1 = 0.1$, and $T = 0.5$)

based valuation model that allows the user to vary the resolution in different parts of the tree. While the binomial tree for American barrier options is not as efficient as it is for standard American options, this adaptive mesh method can provide a more efficient benchmark for comparison with our explicit formulas. The AMM for barrier options with level 5 is used in Tables 1 and 2.

We note that the last column $k^*$ is the optimal policy barrier when $N = 10,000$, and the best constant policy depends, of course, on option parameters such as initial stock price, strike price, upper barrier, and $T_1$.

## 4 Conclusion

This paper studies the valuation problem of American partial barrier option. Because a wide variety of traded options are American type, the problem of valuing American options has been an important topic in financial economics. The literature of American option has proposed good numerical solution methods and anlytic approximations. However, American (partial) barrier options are much more difficult to price. To the best of our knowledge, the literature suggests no approximation formula for American partial barrier options. This paper adopts the barrier approximation method under constant exercise policies, and provides an analytic approximation as the finite sum of bivariate normal distribution functions. Due to this contribution, one can calculate the American partial barier option prices in a simple and speedy way.

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