Mathematical Finance, Vol. 14, No. 2 (April 2004), 163–172

PARETO EQUILIBRIA WITH COHERENT MEASURES OF RISK

DAVID HEATH

Department of Mathematical Sciences, Carnegie Mellon University

HYEJIN KU

Department of Mathematics and Statistics, York University

In this paper, we provide a definition of Pareto equilibrium in terms of risk measures, and present necessary and sufficient conditions for equilibrium in a market with finitely many traders (whom we call "banks") who trade with each other in a financial market. Each bank has a preference relation on random payoffs which is monotonic, complete, transitive, convex, and continuous; we show that this, together with the current position of the bank, leads to a family of valuation measures for the bank. We show that a market is in Pareto equilibrium if and only if there exists a (possibly signed) measure that, for each bank, agrees with a positive convex combination of all valuation measures used by that bank on securities traded by that bank.

KEY WORDS: Pareto equilibrium, coherent measures of risk, preference relation, valuation measure, floor

1. INTRODUCTION

In this paper, we provide a definition of Pareto equilibrium in terms of risk measures, and present necessary and sufficient conditions for equilibrium in a market with finitely many traders (whom we call "banks") who trade with each other in a financial market.

Let X be a set of traded random payoffs. Let a preference relation over X be given by a binary order $\succeq$. We say x is preferred to y if $x \succeq y$. A binary order $\succeq$ on X is monotonic if $x \succeq y$ whenever $x \geq y$, and complete if $y \succeq x$ whenever $x \succeq y$ is not the case. Let G_x denote the "at least as good as x" set $\{y \in X : y \succeq x\}$ and let B_x denote the "at least as bad as x" set $\{y \in X : x \succeq y\}$. A binary relation $\succeq$ is transitive if $x \succeq z$ whenever $x \succeq y$ and $y \succeq z$ for any x, y, and z in X. $\succeq$ is convex if G_x is convex for all $x \in X$. If G_x and B_x are closed for all $x \in X$ then $\succeq$ is continuous.

We assume that each bank has its own preference relation $\succeq_i$ that is monotonic, complete, transitive, convex, and continuous. Following Duffie (1988, p. 36), a continuous preference relation on a set X is represented by a continuous utility function if X is a convex, subset of a separable normed space: There exists a nondecreasing real-valued function U^i on the set X such that

$$x \succeq_i y \iff U^i(x) \geq U^i(y).$$

The first-named author is grateful for support from NSF grant DMS-0139911.

Manuscript received April 2002; final revision received November 2002.

Address correspondence to David Heath, Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213-3890; e-mail: dchcmu@attbi.com.

We assume (e.g. Mossin 1966) that each bank makes trades to improve its utility (preference). Suppose that the random payoff of the current position of a bank is w_0. The bank wants to improve its position by taking a new trade resulting in a preferred one. The set $G_{w_0} = \{y \in X : y \succeq w_0\}$ satisfies the axioms on acceptance sets defined in Artzner et al. (1999), as extended in Artzner et al. (1998), Artzner et al. (2000); for completeness we present this briefly in Section 2. We then have a representation in terms of risk measures associated with an acceptance set. We can describe G_{w_0} by a set $\mathcal{R}$ of "generalized scenarios" or probability measures on Ω as follows:

$$G_{w_0} = \{X : E_R[X] \geq c_R \quad \text{for all } R \in \mathcal{R}\},$$

where c_R are nonpositive constants corresponding to $R \in \mathcal{R}$. For the purposes of this paper we assume $\mathcal{R}$ is a finite set.

When each bank decides whether or not to accept an additional trade, it would like to improve its position in terms of "scenarios" or risk measures. In other words, we may assume that it makes a decision that depends only on its own "level curve" for its utility (preference relation).

Carr, Geman, and Madan (2001) have considered a market under similar assumptions. They considered net trades rather than positions, and they called measures $R \in \mathcal{R}$ "test measures." This corresponds to a change of coordinates in which the new origin is w_0. In these new coordinates, the floor associated with each test measure will typically be different. Therefore, for example, two banks with similar preferences but different initial positions will have different preferences on new trades.

Carr et al. (2001) classified test measures on net trades as *valuation test measures* if the corresponding floor is zero and as *stress test measures* if it is negative. Therefore, this classification changes if the initial position w_0 is changed. (See Section 3 for details.)

The theory of utility maximization postulates that each individual may have a different utility functions. Since each bank has a different initial position as well as a different preference relation, we model each bank as having its own risk measure (set of test measures). As mentioned earlier, we approach a Pareto equilibrium with utility-based arguments and risk measures.

In Section 2, we define Pareto equilibrium in a weak sense and provide a simple condition for equilibrium. In Section 3, we provide a definition of Pareto equilibrium in a strong sense. We give a necessary and sufficient condition for equilibrium in terms of test measures. In Section 4, we consider "incomplete" markets. We assume that for each bank there is a linear space of random payoffs in which that bank can trade. We then find a slightly weaker condition for a Pareto equilibrium than that in Section 3. We present an example which illustrates that our result is tight.

2. RISK MEASURES AND ACCEPTANCE SETS

The material in this section is closely related to the notion of coherent measures of risk as presented in Artzner et al. (1999) and discussed in Artzner et al. (1998).

Let Ω be the set of all possible states; we assume it is a finite set. Let $\mathcal{X}$ denote the set of future net worths—that is, the set of all functions on Ω.

We state axioms for acceptance sets, sets of future net worths that are "acceptable."

AXIOM 2.1. The acceptance set $\mathcal{A}$ is convex.

AXIOM 2.2. If the acceptance set $\mathcal{A}$ contains X and $X \leq Y$, then $\mathcal{A}$ contains Y.

AXIOM 2.3. The acceptance set $\mathcal{A}$ is closed.

AXIOM 2.4. The acceptance set $\mathcal{A}$ and its complement are nonempty.

DEFINITION 2.1. A *risk measure* is a mapping from $\mathcal{X}$ into $\mathbb{R}$.

AXIOM C (Convexity). For all $X, Y \in \mathcal{X}$, and $0 \leq \lambda \leq 1$,

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y).$$

AXIOM T (Translation invariance). For all $X \in \mathcal{X}$ and all real number α,

$$\rho(X + \alpha) = \rho(X) - \alpha.$$

AXIOM M (Monotonicity). For all X and $Y \in \mathcal{X}$ with $X \leq Y$,

$$\rho(Y) \leq \rho(X).$$

DEFINITION 2.2. A risk measure ρ is called *weakly coherent* if it satisfies the above three axioms of convexity, translation invariance, and monotonicity.

A correspondence between acceptance sets and risk measures can be defined as follows.

DEFINITION 2.3. The risk measure associated with the acceptance set $\mathcal{A}$ is the mapping $\rho_{\mathcal{A}}$ from $\mathcal{X}$ to $\mathbb{R}$ defined by

$$\rho_{\mathcal{A}}(X) = \inf\{m \mid m + X \in \mathcal{A}\}.$$

DEFINITION 2.4. The acceptance set associated with a risk measure ρ is the set denoted by $\mathcal{A}_\rho$ and defined by

$$\mathcal{A}_\rho = \{X \in \mathcal{X} \mid \rho(X) \leq 0\}.$$

The next two propositions relate the axioms on acceptance sets and the axioms on risk measures.

PROPOSITION 2.5. *If a risk measure ρ is weakly coherent, then the acceptance set $\mathcal{A}_\rho$ satisfies Axioms 2.1, 2.2, 2.3, 2.4, and $\rho = \rho_{\mathcal{A}_\rho}$.*

PROPOSITION 2.6. *If the set $\mathcal{A}'$ satisfies Axioms 2.1, 2.2, 2.3, and 2.4, the risk measure $\rho_{\mathcal{A}'}$ is weakly coherent and $\mathcal{A}_\rho = \mathcal{A}'$.*

Finally we have the following representation of risk measures by "scenarios" or probability measures and the corresponding floors.

PROPOSITION 2.7. *A risk measure ρ is weakly coherent if and only if there exist a nonempty family $\mathcal{R}$ of probability measures R on Ω and corresponding constants C_R such that*

$$\rho(X) = -\inf_{R \in \mathcal{R}} \{E_R[X] + C_R\}.$$

Proof. The proof is essentially the same as that in Artzner et al. (1999). □

3. THE MODEL

Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_K\}$ be the set of all possible outcomes and X a random payoff (a function on Ω). We assume that there are I banks labeled $i\,(i = 1, \ldots, I)$. We allow each bank to have a different set of test measures. Suppose that the random payoff of the current position of a bank is w_0. Then the set $G_{w_0} = \{y \in X : y \succeq w_0\}$ of the random payoffs preferred to w_0 can be described as

$$G_{w_0} = \{X : E_R[X] \geq c_R \quad \text{for all } R \in \mathcal{R}\}.$$

Consider the set $N_{w_0} = \{\delta : w_0 + \delta \succeq w_0\}$ of net trades resulting in a position preferred to w_0. This set N_{w_0} satisfies Axioms 2.1, 2.2, and 2.3; therefore we get a similar representation by a set of test measures:

$$N_{w_0} = \{\delta : E_R[\delta] \geq c_{R,w_0} \text{ for all } R \in \mathcal{R}\}.$$

Carr et al. (2001) called each c_{R,w_0} a "floor." It is easy to see that if $w_0 \succeq z_0$ and $z_0 \succeq w_0$ then the set $\mathcal{R}$ of test measures in the representation is the same, but c_{R,z_0} may be different from c_{R,w_0}. The floor associated with each test measure depends on the current position of each bank. We call test measures with zero floors *valuation measures*. The set of valuation measures determined by the bank's preference and current position contributes to valuation of additional trades of the bank at that position. We use $\mathcal{R}_i = \{R_{i1}, R_{i2}, \ldots, R_{in_i}\}$ to denote the (assumed finite) set of valuation measures of bank i.

In this section, we define Pareto equilibrium in a weak sense as the following: A market is in weak-sense Pareto equilibrium if there is no new "deal (or trade)" between any pair of banks that can increase one bank's preference and at the same time not reduce the other bank's preference.

DEFINITION 3.1. Two banks, say Bank i_1 and Bank i_2, are *in Pareto equilibrium* if there is no net trade X that satisfies

$$\begin{aligned} E_{R_{i_1 j}}[X] &\geq 0 \qquad (j = 1, \ldots, n_{i_1}), \\ E_{R_{i_2 j}}[-X] &\geq 0 \qquad (j = 1, \ldots, n_{i_2}), \end{aligned}$$

and at least one of the above inequalities is strict.

In weak-sense Pareto equilibrium, the following: Linear Programming Problem (LP)

$$\begin{aligned} \text{Max} \quad & \left(\sum_{j=1}^{n_{i_1}} E_{R_{i_1 j}}[X]\right) - \left(\sum_{j=1}^{n_{i_2}} E_{R_{i_2 j}}[X]\right) \\ \text{subject to} \; & -E_{R_{i_1 j}}[X] \leq 0 \qquad (j = 1, \ldots, n_{i_1}), \\ & E_{R_{i_2 j}}[X] \leq 0 \qquad (j = 1, \ldots, n_{i_2}), \end{aligned}$$

is bounded. (In that case, the optimal value equals 0.) We can write this LP as

$$\begin{aligned} \text{Max} \quad & \sum_j \sum_k R_{i_1 j}(\omega_k) X(\omega_k) - \sum_j \sum_k R_{i_2 j}(\omega_k) X(\omega_k) \\ \text{subject to} \quad & A_1 X \leq b \\ & A_2 X \leq b, \end{aligned}$$

where

$$A_1 = \begin{pmatrix} -R_{i_1 1}(\omega_1) & -R_{i_1 1}(\omega_2) & \dots & -R_{i_1 1}(\omega_K) \\ -R_{i_1 2}(\omega_1) & -R_{i_1 2}(\omega_2) & \dots & -R_{i_1 2}(\omega_K) \\ \vdots & \vdots & \ddots & \vdots \\ -R_{i_1 n_{i_1}}(\omega_1) & -R_{i_1 n_{i_1}}(\omega_2) & \dots & -R_{i_1 n_{i_1}}(\omega_K) \end{pmatrix},$$

$$A_2 = \begin{pmatrix} R_{i_2 1}(\omega_1) & R_{i_2 1}(\omega_2) & \dots & R_{i_2 1}(\omega_K) \\ R_{i_2 2}(\omega_1) & R_{i_2 2}(\omega_2) & \dots & R_{i_2 2}(\omega_K) \\ \vdots & \vdots & \ddots & \vdots \\ R_{i_2 n_{i_2}}(\omega_1) & R_{i_2 n_{i_2}}(\omega_2) & \dots & R_{i_2 n_{i_2}}(\omega_K) \end{pmatrix},$$

and $b = (0, \dots, 0)^t$.

Let $Y = (y_1, y_2, \dots, y_{n_{i_1}+n_{i_2}})$ represent the corresponding dual variables. The dual problem can be written as

$$\begin{aligned} \text{Min} \quad & Yb \\ \text{subject to} \quad & YA_1 = C \\ & YA_2 = C, \end{aligned}$$

where $C = (\sum_{j=1}^{n_{i_1}} R_{i_1 j}(\omega_1) - \sum_{j=1}^{n_{i_2}} R_{i_2 j}(\omega_1), \dots, \sum_{j=1}^{n_{i_1}} R_{i_1 j}(\omega_K) - \sum_{j=1}^{n_{i_2}} R_{i_2 j}(\omega_K))$. Note that the dual constraints would be equality constraints if the primal variables were unrestricted in sign.

By the Dual Theorem (e.g., see Winston 1991, p. 276) the dual problem is feasible if and only if the primal LP is bounded. The feasibility of the dual problem means that there exists at least one $(n_{i_1} + n_{i_2})$-dimensional vector $(y_1, y_2, \dots, y_{n_{i_1}+n_{i_2}})$ that is nonnegative and satisfies, for each ω_k $(k = 1, \dots, K)$,

$$-\sum_{j=1}^{n_{i_1}} R_{i_1 j}(\omega_k) y_j + \sum_{j=1}^{n_{i_2}} R_{i_2 j}(\omega_k) y_{n_{i_1}+j} = \sum_{j=1}^{n_{i_1}} R_{i_1 j}(\omega_k) - \sum_{j=1}^{n_{i_2}} R_{i_2 j}(\omega_k)$$

or, equivalently,

$$\sum_{j=1}^{n_{i_1}} R_{i_1 j}(\omega_k)(1 + y_j) = \sum_{j=1}^{n_{i_2}} R_{i_2 j}(\omega_k)(1 + y_{n_{i_1}+j}). \tag{3.1}$$

Summing up with respect to k on both sides, we get

$$\sum_{j=1}^{n_{i_1}} (1 + y_j) = \sum_{j=1}^{n_{i_2}} (1 + y_{n_{i_1}+j}).$$

Let $\bar{y}$ denote the value of this sum and let $\alpha_j = \frac{(1+y_j)}{\bar{y}}$ $(j = 1, \dots, n_{i_1})$ and $\beta_j = \frac{(1+y_{n_{i_1}+j})}{\bar{y}}$ $(j = 1, \dots, n_{i_2})$. Then we can rewrite (3.1) as

$$\sum_{j=1}^{n_{i_1}} \alpha_j R_{i_1 j}(\omega_k) = \sum_{j=1}^{n_{i_2}} \beta_j R_{i_2 j}(\omega_k),$$

where $0 < \alpha_j, \beta_j < 1$, and $\sum_{j=1}^{n_{i_1}} \alpha_j = \sum_{j=1}^{n_{i_2}} \beta_j = 1$.

Therefore, we have the following propositions.

PROPOSITION 3.2. *A market is in Pareto equilibrium in the weak sense if and only if for any pair of banks, Bank i_1 and Bank i_2 $(1 \leq i_1, i_2 \leq I)$, there exist α_j, β_j* such that $0 < \alpha_j, \beta_j < 1$, $\sum_{j=1}^{n_1} \alpha_j = \sum_{j=1}^{n_2} \beta_j = 1$, *and*

$$\sum_{j=1}^{n_{i_1}} \alpha_j R_{i_1 j}(\omega) = \sum_{j=1}^{n_{i_2}} \beta_j R_{i_2 j}(\omega) \quad \textit{for all } \omega \in \Omega.$$

In other words, the positive convex hulls of any pair of $\{\mathcal{R}_i : 1 \leq i \leq I\}$ intersect each other, where we define the positive convex hull of a finite set $A = \{a_1, a_2, \ldots, a_n\}$ as $\{\sum_{i=1}^n \lambda_i a_i : \sum_{i=1}^n \lambda_i = 1, \lambda_i > 0$ for all $i\}$.

4. PARETO EQUILIBRIUM IN A MARKET

In this section, we define the notion of Pareto equilibrium among I banks in a strong sense.

DEFINITION 4.1. A market consisting of I banks is *in Pareto equilibrium* in the strong sense if there is no deal $X = (X_1, X_2, \ldots, X_I)$ in the market that satisfies $\sum_{i=1}^I X_i = 0$,

$$E_{R_{ij}}[X_i] \geq 0$$

for each i, j $(i = 1, \ldots, I; j = 1, \ldots, n_i)$, and at least one of the above inequalities is strict.

Clearly, if a market is in Pareto equilibrium in the strong sense, it is in equilibrium in the weak sense. From now on, we use the notion of Pareto equilibrium in the strong sense with no specification.

PROPOSITION 4.2. *A market consisting of I banks is in Pareto equilibrium if and only if*

$$\bigcap_{i=0}^{I} co^+\{\mathcal{R}_i\} \neq \emptyset,$$

where $co^+\{\mathcal{R}_i\} = \{\sum_{j=1}^{n_1} \alpha_{ij} R_{ij} : 0 < \alpha_{ij} < 1, \sum_{j=1}^{n_1} \alpha_{ij} = 1$ *and* $R_{ij} \in \mathcal{R}_i\}$; *that is, the intersection of all positive convex hulls of* $\mathcal{R}_i$ $(i = 1, \ldots, I)$ *is nonempty.*

Proof. Suppose that the market is in Pareto equilibrium. Consider the following LP:

$$\begin{aligned} \text{Max} \quad & \sum_i \sum_j E_{R_{ij}}[X_i] \\ \text{subject to} \quad & E_{R_{ij}}[X_i] \geq 0 \quad \text{for all } i, j\ (i = 1, \ldots, I, j = 1, \ldots, n_i) \\ & \sum_{i=1}^I X_i = 0, \end{aligned}$$

or

$$\begin{aligned} \text{Max} \quad & \sum_i \sum_j \sum_k R_{ij}(\omega_k) X_i(\omega_k) \\ \text{subject to} \quad & A^u X \leq 0 \\ & A^l X = 0, \end{aligned} \tag{4.1}$$

where the matrix A^u is a block diagonal matrix

$$\begin{pmatrix} A_1^u & 0 & \dots & 0 \\ 0 & A_2^u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_I^u \end{pmatrix},$$

whose ith block A_i^u is of the form

$$\begin{pmatrix} -R_{i1}(\omega_1) & -R_{i1}(\omega_2) & \dots & -R_{i1}(\omega_K) \\ -R_{i2}(\omega_1) & -R_{i2}(\omega_2) & \dots & -R_{i2}(\omega_K) \\ \vdots & \vdots & \ddots & \vdots \\ -R_{in_i}(\omega_1) & -R_{in_i}(\omega_2) & \dots & -R_{in_i}(\omega_K) \end{pmatrix},$$

the matrix A^l consists of K row vectors, each vector A_j^l has $I \times K$ elements and is of the form

$$\big(0, \cdot\cdot, 1_j, \cdot\cdot, 0; 0, \cdot\cdot, 1_j, \cdot\cdot, 0; \cdots; 0, \cdot\cdot, 1_j, \cdot\cdot, 0\big),$$

and $X = (X_1(\omega_1), X_1(\omega_2), \dots, X_1(\omega_K); X_2(\omega_1), X_2(\omega_2), \dots, X_2(\omega_K); \dots; X_I(\omega_1), X_I(\omega_2), \dots, X_I(\omega_K))^t$ is a vector of the primal variables.

We use an argument similar to that in Section 3. Let $y_{11}, y_{12}, \dots, y_{1n_1}; y_{21}, y_{22}, \dots, y_{2n_2}; \dots; y_{I1}, y_{I2}, \dots, y_{In_I}; z_1, z_2, \dots, z_K$ be the dual variables. Then the dual constraints of the above LP can be written as

$$-\sum_{j=1}^{n_i} R_{ij}(\omega_k) y_{ij} + z_k = \sum_{j=1}^{n_i} R_{ij}(\omega_k),$$

where $y_{ij} \geq 0$, z_k unrestricted in sign. Rewriting this as

$$\sum_{j=1}^{n_i} (1 + y_{ij}) R_{ij}(\omega_k) = z_k,$$

and summing up over k, we have

$$\sum_{j=1}^{n_i} (1 + y_{ij}) = \sum_{k=1}^{K} z_k.$$

Let $\bar{z}$ denote this sum. Therefore the boundedness of LP implies that there exist y_{ij} and z_k such that

$$\sum_{j=1}^{n_i} \frac{1 + y_{ij}}{\bar{z}} R_{ij}(\omega_k) = \frac{z_k}{\bar{z}}.$$

The right-hand side is independent of i. Therefore, letting $\alpha_{ij} = \frac{1 + y_{ij}}{\bar{z}}$ $(j = 1, \dots, n_i)$, there exist α_{ij} and z_k such that

$$\sum_{j=1}^{n_i} \alpha_{ij} R_{ij}(\omega_k) = \frac{z_k}{\bar{z}},$$

where $0 < \alpha_{ij} < 1$ and $\sum_{j=1}^{n_i} \alpha_{ij} = 1$ for each i. Hence, we get the desired result.

Conversely, suppose that there exist β_{ij} and γ_k such that $0 < \beta_{ij} < 1$, $\sum_{j=1}^{n_1} \beta_{ij} = 1$ for each i, $\sum_{k=1}^{K} \gamma_k = 1$, and

$$\gamma_k = \sum_{j=1}^{n_i} \beta_{ij} R_{ij}(\omega_k).$$

Then there exists a positive constant N such that $N\beta_{ij} \geq 1$. Set $y_{ij} = N\beta_{ij} - 1 \geq 0$ and $z_k = N\gamma_k$. It is easy to check that these y_{ij} and z_k are feasible solutions to the above dual problem. By using the Dual Theorem again, the primal LP is bounded, which means the market is in Pareto equilibrium.

5. "INCOMPLETE" MARKETS

So far, we implicitly allowed all banks to trade any random variables they want. We now suppose that each bank trades only those payoffs in some linear space M_i. Let X_i be a random variable on Ω and $X_i \in M_i$. We can append constraints $BX = 0$ to (4.1) where B is a block diagonal matrix:

$$B = \begin{pmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_I \end{pmatrix}.$$

We assume that M_i is a $(K - d_i)$ dimensional space, the ith block B_i is a $(d_i \times K)$ matrix, and its row vectors, $B_{i1}, \dots, B_{id_i}$, are linearly independent and orthogonal to M_i.

We introduce new dual variables $\bar{y}_{11}, \dots, \bar{y}_{1d_1}; \bar{y}_{21}, \dots, \bar{y}_{2n_2}; \dots; \bar{y}_{I1}, \dots, \bar{y}_{Id_I}$. Then the dual constraints of the above LP are written as

$$-\sum_{j=1}^{n_i} R_{ij}(\omega_k) y_{ij} + \sum_{j=1}^{n_i} B_{ij}(\omega_k)\bar{y}_{ij} + z_k = \sum_{j=1}^{n_i} R_{ij}(\omega_k),$$

where $y_{ij} \geq 0$ and $\bar{y}_{ij}$, z_k unrestricted in sign. Summing over k,

$$\sum_{j=1}^{n_i}(1 + y_{ij}) = \sum_{k=1}^{K} \left(\sum_{j=1}^{d_i} B_{ij}(\omega_k)\bar{y}_{ij} + z_k \right).$$

Let $\bar{z}$ denote this sum, and set

$$\mu_i(\omega_k) = \sum_{j=1}^{n_i} \frac{1 + y_{ij}}{\bar{z}} R_{ij}(\omega_k) = \sum_{j=1}^{d_i} \frac{\bar{y}_{ij}}{\bar{z}} B_{ij}(\omega_k) + \frac{z_k}{\bar{z}}$$

and

$$\bar{\mu}(\omega_k) = \mu_i(\omega_k) - \sum_{j=1}^{d_i} \frac{\bar{y}_{ij}}{\bar{z}} B_{ij}(\omega_k).$$

Since B_{ij} is orthogonal to M_i for each i, j,

$$E_{\bar{\mu}}[X_i] = E_{\mu_i}[X_i]$$

if $X_i \in M_i$.

PROPOSITION 5.1. *Suppose that a market consists of I banks and Bank i can trade X_i for $X_i \in M_i$. The market is in Pareto equilibrium if and only if there exists a (possibly signed) measure $\bar{\mu}$ such that for each i, there is a probability measure $\mu_i \in co^+\{\mathcal{R}_i\}$ for which*

$$E_{\bar{\mu}}[X_i] = E_{\mu_i}[X_i]$$

for all $X_i \in M_i$.

We present a simple example illustrating the case in which we can find only a signed (not positive) measure satisfying the conclusion of Proposition 5.1, even though a market is in Pareto equilibrium.

EXAMPLE 5.2. Let Ω denote the set $\{\omega_1, \omega_2, \omega_3\}$. Suppose that Bank 1 trades only securities in the linear space $M_1 = \{(x, -x, y) : x, y \in \mathbb{R}\}$ and Bank 2 trades only securities in $M_2 = \{(u, v, -v) : u, v \in \mathbb{R}\}$. Then the vectors $B_1 = (1, 1, 0)$, $B_2 = (0, 1, 1)$ are orthogonal to M_1, M_2 respectively, and $M_1 \cap M_2 = \{(x, -x, x) : x \in \mathbb{R}\}$.

We assume that Bank 1 has a valuation measure $R_1 = (R_1(\omega_1), R_1(\omega_2), R_1(\omega_3)) = (1, 0, 0)$ and Bank 2 has a valuation measure $R_2 = (R_2(\omega_1), R_2(\omega_2), R_2(\omega_3)) = (0, 0, 1)$. The market consisting of Bank 1 and Bank 2 is then in Pareto equilibrium because there is no trade $X \in M_1 \cap M_2$ such that $E_{R_1}[X] \geq 0$ and $E_{R_2}[X] \leq 0$ with at least one inequality being strict.

Let $\bar{\mu} = (\bar{\mu}(\omega_1), \bar{\mu}(\omega_2), \bar{\mu}(\omega_3))$ be a measure such that $E_{\bar{\mu}}[X_1] = E_{\mu_1}[X_1] = E_{R_1}[X_1]$ and $E_{\bar{\mu}}[X_2] = E_{\mu_2}[X_2] = E_{R_2}[X_2]$ for all $X_1 \in M_1$ and $X_2 \in M_2$.

Since $E_{\bar{\mu}}[X_1] = E_{R_1}[X_1]$ for any $X_1 \in M_1$, we have

$$\bar{\mu}(\omega_1) - \bar{\mu}(\omega_2) = 1, \qquad \bar{\mu}(\omega_3) = 0.$$

On the other hand, since $E_{\bar{\mu}}[X_2] = E_{R_2}[X_2]$ for any $X_2 \in M_2$, we have

$$\bar{\mu}(\omega_2) - \bar{\mu}(\omega_3) = -1, \qquad \bar{\mu}(\omega_1) = 0.$$

Therefore, a signed measure $\bar{\mu} = (0, -1, 0)$ is the only possible measure.

The following corollary is an immediate consequence of Proposition 5.1. If there is at least one bank that can trade all securities in a market, then a pricing measure $\bar{\mu}$ in Proposition 5.1 has to be a probability measure.

COROLLARY 5.3. *Suppose that there exists a bank that can trade any random variable in a market. ($M_i = \mathbb{R}^K$ for some i.) Then the market is in Pareto equilibrium if and only if there exists a probability measure $\bar{\mu}$ such that for each i, there exists a probability measure $\mu_i \in co^+\{\mathcal{R}_i\}$ for which*

$$E_{\bar{\mu}}[X_i] = E_{\mu_i}[X_i]$$

for all $X_i \in M_i$.

If all banks in a market can trade only those securities in the same subspace, then we get the same result as in Proposition 4.2.

COROLLARY 5.4. *Suppose $M_1 = M_2 = \cdots = M_I$. Then the market is in Pareto equilibrium if and only if*

$$\bigcap_{i=0}^{I} co^+\{\mathcal{R}_i\} \neq \emptyset.$$

REFERENCES

ARTZNER, P., F. DELBAEN, J.-M. EBER, and D. HEATH (1998): *Proceedings of the ICBI 5th Annual Conference on Risk Management*, Geneva, Switzerland.

ARTZNER, P., F. DELBAEN, J.-M. EBER, and D. HEATH (1999): Coherent Measures of Risk, *Math. Finance* 9, 203–228.

ARTZNER, P., F. DELBAEN, J.-M. EBER, and D. HEATH (2000): *Proceedings of the The First World Congress of the Bachelier Finance Society*, Paris.

CARR, P., H. GEMAN, and D. MADAN (2001): Pricing and Hedging in Incomplete Markets, *J. Financial Econ*. 62, 131–167.

DUFFIE, D. (1988): *Security Markets*. New York: Academic Press.

MOSSIN, J. (1966): Equilibrium in a Capital Asset Market, *Econometrics* 34, 768–783.

WINSTON, W. L. (1991): *Operations Research: Applications and Algorithms*, 2nd ed. PWS-Kent.