

## VALUATION AND HEDGING OF OPTIONS WITH GENERAL PAYOFF UNDER TRANSACTIONS COSTS

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ABSTRACT. We present the pricing and hedging method for options with general payoffs in the presence of transaction costs. The convexity of the payoff function - gamma of the options - is an important issue under transaction costs. When the payoff function is convex, Leland-style pricing and hedging method still works. However, if the payoff function is of general form, additional assumptions on the size of transaction costs or of the hedging interval are needed. We do not assume that the payoff is convex as in Leland [11] and the value of the Leland number is less (bigger) than 1 as in Hoggard et al. [10], Avellaneda and Parás [1]. We focus on generally recognized asymmetry between the option sellers and buyers. We decompose an option with general payoff into difference of two options each of which has a convex payoff. This method is consistent with a scheme of separating out the seller's and buyer's position of an option. In this paper, we first present a simple linear valuation method of general payoff options, and also propose in the last section more efficient hedging scheme which costs less to hedge options.

### 1. Introduction

The celebrated Black-Scholes partial differential equation is a general method of valuing European style contingent claims. Since this partial differential equation can be solved for *any* sufficiently regular boundary condition, this means that any contingent claim with any form of payoff can be valued. This mathematical fact gives the fundamental tenet of option pricing theory. In particular, the option pricing mechanism is indifferent to buyers or sellers. The buyer and seller can each engage

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in a continuous delta hedging to arrive at the same result, at least in theory. However these results are valid only under the assumption that there are no transaction costs.

When the transaction costs are taken into account, a radically different picture emerges. The sources of transaction costs are manifold: besides the usual fees paid to various intermediaries, the bid-ask spread, slippage and other market impact costs could quickly add up to a significant sum. In the presence of transaction costs, the usual Black-Scholes style dynamic hedging is no longer riskless, and a lot of research effort has attempted to deal with this problem.

An important and elegant breakthrough was achieved by Leland [11]. He introduced a method of pricing a call option from the seller's viewpoint in the presence of transaction costs. His main idea is to increase the volatility in the Black-Scholes PDE to offset the increased risk of the seller. He gave his argument for the call option. However, it works as long as the payoff function is convex. Boyle and Vorst [4] derived an option pricing formula in the Cox-Ross-Rubinstein binomial model with transaction costs. They constructed a replicating portfolio in each trading interval and their result has strong similarity with that of Leland's model. With the same discrete-time approach, Bensaid et al. [2] considered optimal hedging strategies whose terminal values are allowed to dominate the final payoff of derivative at maturity.

Let  $\sigma$  be the volatility constant in the Black-Scholes-Merton formula and let  $\Delta t$  be the hedging interval and  $k$  be the proportional transaction cost ratio. Leland introduced a new increased volatility

$$\sigma_A = \sigma\sqrt{1 + A},$$

where

$$A = \sqrt{\frac{2}{\pi}} \frac{k}{\sigma\sqrt{\Delta t}}$$

is the so-called Leland number. He proved that if the seller charges the premium in the form of increased volatility  $\sigma_A$ , the hedging error can be made to go to zero as  $\Delta t$  goes to zero. In his argument, the convexity assumption is crucially used to cancel off certain dominant terms. Because of this, Leland's argument cannot be carried over to the buyer of a call option or to the seller of an option whose payoff function is concave. In this case, to similarly cancel off dominant terms,  $\sigma_A$  has to be modified to

$$\sigma_A = \sigma\sqrt{1 - A}.$$

However, when  $k$  is large enough or the hedging interval  $\Delta t$  is small enough, the Leland number  $A$  becomes greater than one, in which case this modified  $\sigma_A$  no longer makes sense. This means that one cannot hope to make the hedging error become zero by employing more frequent hedging schedules. This fact necessitates a rather sparse hedging schedule. However, this is not so bad when we consider the limited risk the buyer of call option actually takes. Moreover, its mean hedging error is zero. We will come back to this point later.

When the payoff is of general form, i.e., when it is neither convex nor concave, the above Leland style approach does not work. However, by employing Leland's argument and demanding the expected hedging error to be zero, Hoggard et al. [10] have introduced a method based on the following nonlinear PDE

$$C_t + \frac{1}{2}\sigma(\Gamma)^2 S^2 C_{SS} - rC + rSC_S = 0,$$

where  $\sigma(\Gamma)$  is defined as

$$\sigma(\Gamma) = \sigma\sqrt{1 + A\text{sign}(\Gamma)}.$$

This amounts to switching between the seller's and the buyer's position according to the sign of  $\Gamma = C_{SS}$ . Since the convexity is no longer guaranteed, this method is restricted to the case with small transaction costs, i.e.,  $A < 1$ . Thus again, one cannot take the hedging interval to zero.

As an extension of Leland's, Whalley and Wilmott [12] set up a model which does not necessarily re hedge at every hedging time. That is, if there is no or little change in delta then the option portfolio is not rebalanced. They derived for the option value

$$C_t + \frac{1}{2}\sigma^2 S^2 C_{SS} - rC + rSC_S = \frac{k}{\sqrt{\epsilon}}\sigma^2 S^4 C_{SS}^2.$$

Here, when the delta moves out of line by more than  $\epsilon/S$ , the portfolio should be rehedge. Therefore, an option is valued according to a choice of the bandwidth  $\epsilon$ . They obtained the above equation with a general form of costs which incorporates a fixed cost, a cost proportional to volume traded and a cost proportional to value traded. But there is no significant difference in the derivation procedure.

When transaction costs are large, meaning  $A > 1$ , the above Hoggard-Whalley-Wilmott method cannot be used. To deal with this case, Avelaneda and Parás [1] have introduced nonlinear obstacle problem. Their approach is to divide the range of terminal stock prices into the intervals

such that the payoff function restricted to each interval is convex and then, to solve the several obstacle problems. In fact, solving the given obstacle problem is equal to finding the minimal value among those which dominate the payoff. However, their obstacle problem is set up in such a way that the seller's price in certain cases has to be unreasonably high. We will come back to this point later in connection with our method.

Another main approach was introduced by Hodges and Neuberger [9] and modified by Davis et al. [6]. They valued the option by a more general management approach. They employed a utility function, and found the strategy which is optimal in the sense of the maximization of the expected utility. They found that there is a region around the Black-Scholes delta where no transaction is made. But the computation speed and dependency on an investor's utility function are depicted as a weak point.

As all these previous works amply demonstrate, the convexity of the payoff function is an important issue when there are transaction costs. For example, the seller of an option with a convex payoff function has a short-gamma position. This situation is similar to that of the buyer of an option whose payoff function is concave, in which case the buyer's position is also short-gamma. In either case, the option valuation and hedging can be done by the Leland style method. On the other hand, the seller of an option with concave payoff function or the buyer of an option with convex payoff function has a long-gamma position, which is harder to deal with because of the reason outlined above. Consider, for instance, an option with convex payoff function. The buyer of this option has limited risk, namely, the time value, if he/she engages in a static hedging at the time of purchase. On the other hand, the seller of the same option is faced with *possible unlimited loss*, unless he/she engages in a proper delta hedging strategy. This economic asymmetry between the buyer and seller becomes an acute problem when there are significant transaction costs.

This fundamental asymmetry between the buyer and seller has led us to a scheme of separating out the inherent risks associated with the two positions mixed in the general payoff function. Our idea of separating out the risks is to decompose an option with general payoff function into a difference of two options each of which has a convex payoff function. This amounts to separating out the buyer's and seller's risks mixed in the general form of the option, and then treating each risk differently. This decomposition method is consistent with the generally recognized

risk asymmetry between the seller and the buyer in the presence of transaction costs. The basic strategy of our method goes as follows: Suppose an option has a payoff function  $\varphi = \varphi_1 - \varphi_2$ , where  $\varphi_1$  and  $\varphi_2$  are convex functions according to our decomposition scheme. We treat the option somewhat like a portfolio of options and we value it from the option seller's standpoint. From this angle, the portion of the option with the payoff  $\varphi_2$  is like a position (long gamma) that entails limited risk. We thus fix its price  $V^b$  assuming that we will employ a static or sparse hedging strategy. This is consistent with the risk profile of the buyer. Next, we fix the price  $V^s$  for the payoff  $\varphi_1$ . Since we compute it from the seller's viewpoint and since  $\varphi_1$  is convex, the Leland style method of frequent hedging can be employed. The resulting seller's price of the whole option is then  $V = V^s - V^b$ .

In Section 4, our solution is compared with that of Hoggard et al. [10]. We found that two solutions more or less coincide. Our important consequence of this is that we can write down an explicit formula for the option price if the transaction costs are low and the payoff function is piecewise linear. This is of practical value, because it gives a fast, reliable answer without solving *nonlinear* partial differential equation.

Moreover, two known methods, one due to Hoggard et al. [10], and the other due to Avellaneda and Parás [1], deal with cases when  $A < 1$  and  $A > 1$  respectively; and they cannot be mixed. Thus if the hedging interval  $\Delta t$  gets sufficiently smaller or larger, one has to switch from one method to the other. On the other hand, our method works for either case. This, we feel, is another added advantage of our method.

Finally, we propose another valuation and hedging method in Section 5, which hedges short position for the whole option. Actually, we don't need to pay the transaction costs for the stock trades which can be offset between the short position for  $\varphi_1$  and long position for  $\varphi_2$ . Hence, this modified method reduces the transaction costs and gives more realistic pricing.

## 2. Previous works

Leland has introduced a seminal argument on how to handle the transaction costs from the call option seller's standpoint. Let us first go over his argument. Divide the time period  $[0, T]$  in question into  $N$  equal subintervals. Suppose that over the  $j$ th time interval  $[(j-1)\Delta t, j\Delta t]$ , a

stock price  $S$  satisfies the following discrete equation

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma w \sqrt{\Delta t}$$

where  $w$  is normally distributed random variable with mean zero and variance one.

Consider a portfolio  $P$  consisting of  $N$  shares of stock and  $B$  dollars of the risk-free bond at each discrete time. We revise the portfolio at the beginning of each interval. (The length of the interval  $\Delta t$  would be the revision period.) We define the replicating strategy

$$\begin{aligned} N &= C_S \\ B &= C - C_S S. \end{aligned}$$

In the Black-Scholes-Merton world, delta hedging makes the risk tend to zero. But in the model with transaction costs, we cannot eliminate all the risk. Therefore, the hedging strategy always generates errors and we need to measure the difference  $\Delta H$  between the change in value of the replicating portfolio and the option over the period  $[(j-1)\Delta t, j\Delta t]$ . Over the  $j$ th interval  $[(j-1)\Delta t, j\Delta t]$ , the return of the portfolio  $P$  will be

$$\begin{aligned} \Delta P &= N\Delta S + Br\Delta t + O(\Delta t^2) \\ &= C_S\Delta S + (C - C_S S)r\Delta t + O(\Delta t^2). \end{aligned}$$

Note that  $O(\Delta t^2)$  comes from the continuous compounding of interest. The change in value of a call option  $C$  over the same interval will be

$$\begin{aligned} \Delta C &= C(S + \Delta S, t + \Delta t) - C(S, t) \\ &= C_S\Delta S + C_t\Delta t + \frac{1}{2}C_{SS}(\Delta S)^2 + O(\Delta t^{\frac{3}{2}}), \end{aligned}$$

The number of assets bought or sold is

$$\begin{aligned} \Delta N &= C_S(S + \Delta S, t + \Delta t) - C_S(S, t) \\ &= C_{SS}(S, t)\Delta S + C_{St}(S, t)\Delta t + \frac{1}{2}C_{SSS}(S, t)(\Delta S)^2 + O(\Delta t^{\frac{3}{2}}) \\ &= C_{SS}(S, t)\Delta S + O(\Delta t^{\frac{3}{2}}). \end{aligned}$$

The transaction costs in this interval will be

$$\begin{aligned} \frac{1}{2}k(S + \Delta S)|\Delta N| &= \frac{1}{2}k(S + \Delta S)|C_{SS}\Delta S| + O(\Delta t^{\frac{3}{2}}) \\ &= \frac{1}{2}kS^2 \left| C_{SS} \frac{\Delta S}{S} \right| + O(\Delta t^{\frac{3}{2}}), \end{aligned}$$

Considering the hedging error  $\Delta H$ , we get

$$\begin{aligned}\Delta H &= \Delta P - \Delta C - \text{Transaction Costs} \\ &= (C - C_S S)r\Delta t - C_t\Delta t - \frac{1}{2}C_{SS}(\Delta S)^2 - \frac{1}{2}kS^2 \left| C_{SS} \frac{\Delta S}{S} \right| \\ &\quad + O\left(\Delta t^{\frac{3}{2}}\right) \\ &= \left( rC - rSC_S - C_t - \frac{1}{2}S^2C_{SS} \left( \sigma^2w^2 + \frac{k\sigma|w|}{\sqrt{\Delta t}} \text{sign}(C_{SS}) \right) \right) \Delta t \\ &\quad + O\left(\Delta t^{\frac{3}{2}}\right).\end{aligned}$$

If we demand this hedging error to have zero expectation, we arrive at the following Hoggard-Whalley-Wilmott equation

$$C_t + \frac{1}{2}\sigma_A^2 S^2 C_{SS} - rC + rSC_S = 0$$

where

$$\sigma_A = \sigma\sqrt{1 + A \text{sign}(C_{SS})}.$$

Hoggard et al. [10] used this nonlinear partial differential equation to value options with general payoff functions. Although this equation is derived to make the mean hedging error zero, it performs well when the Leland number is small, i.e., the transaction costs are small and the hedging interval is large enough. Obviously this equation becomes an ill-posed parabolic equation when the Leland number is bigger than one, and thus is not usable in that case.

When the payoff function is convex, the above equation becomes linear since  $C$  itself is convex. In this case,  $C$  satisfies the partial differential equation

$$C_t + \frac{1}{2}\sigma_A^2 S^2 C_{SS} - rC + rSC_S = 0,$$

where  $\sigma_A$  is given by

$$\sigma_A = \sigma\sqrt{1 + A}.$$

Furthermore, Leland was able to show that the total hedging error over  $[0, T]$  tends to zero almost surely, using the law of large numbers. We review more his work in Section 3.1.

### 3. Fundamental asymmetry between buyers and sellers

When there are transaction costs, the convexity-the gamma of the option-becomes a very important issue. When one sells an option with a convex payoff function, he or she has a short gamma position. This

situation is similar to buying an option with a concave payoff function in the sense that in either case the trade has a short gamma position, which has an important bearing on the valuation and hedging of an option. On the other hand, when one sells an option with concave payoff, the seller has a long gamma position. Thus, from the standpoint of our subsequent discussion, the seller of an option with a concave payoff function has a financial situation similar to the buyer of an option with a convex payoff function.

Since the simple term “seller” or “buyer” may likewise cause confusion, we have to agree on the terminology for the sake of clarity. In this paper, unless stated explicitly otherwise, we mean by a seller a person who sells an option whose payoff function is convex, for instance, a call or put option. Thus a seller has a short gamma position; similarly, a buyer is one who buys an option with convex payoff function; thus a buyer has a long gamma position.

As will be shown below, we can always decompose an options with general payoff to a portfolio consisting of one long position and one short of options each of which has a convex payoff. In this sense, our artificial restriction on the term “buyer” or “seller” causes no loss of generality.

### 3.1. Seller’s viewpoint

In this subsection, we discuss more on Leland’s argument for a call option which is easily extended for options with convex payoff function. As pointed out in the previous section, when the payoff function is convex, the option seller has a linear equation for the option value in the presence of transaction costs. We rewrite the transaction costs in the time interval  $[(j-1)\Delta t, j\Delta t]$ ,

$$\begin{aligned} \frac{1}{2}k(S + \Delta S)|\Delta N| &= \frac{1}{2}k(S + \Delta S)|C_{SS}\Delta S| + O\left(\Delta t^{\frac{3}{2}}\right) \\ &= \frac{1}{2}kS^2C_{SS}\left|\frac{\Delta S}{S}\right| + O\left(\Delta t^{\frac{3}{2}}\right), \end{aligned}$$

using the same notations as in Section 2. The last equality is due to the fact that the solution is convex, which is inherited from the convexity assumption of the payoff function. Then the hedging error will be

$$\begin{aligned} \Delta H &= \Delta P - \Delta C - \text{Transaction Costs} \\ &= \left( rC - rSC_S - C_t - \frac{1}{2}S^2C_{SS}\left(\sigma^2w^2 + \frac{k\sigma|w|}{\sqrt{\Delta t}}\right) \right) \Delta t \\ &\quad + O\left(\Delta t^{\frac{3}{2}}\right). \end{aligned}$$



Thus we get the following partial differential equation that should be satisfied to make the expected hedging error zero,

$$C_t + \frac{1}{2}\sigma_A^2 S^2 C_{SS} - rC + rSC_S = 0,$$

where  $\sigma_A$  is given by

$$\sigma_A = \sigma\sqrt{1+A}.$$

Also, substituting for  $(C - C_S S)r\Delta t$  in hedging error gives

$$\begin{aligned} \Delta H &= \frac{1}{2}\sigma_A^2 S^2 C_{SS}\Delta t - \frac{1}{2}C_{SS}(\Delta S)^2 - \frac{1}{2}kC_{SS}S^2 \left| \frac{\Delta S}{S} \right| + O\left(\Delta t^{\frac{3}{2}}\right) \\ &= \frac{1}{2}S^2 C_{ss} \left( \sigma_A^2 \Delta t - \sigma^2 w^2 \Delta t - k\sigma|w|\sqrt{\Delta t} \right) + O\left(\Delta t^{\frac{3}{2}}\right) \\ &= \frac{1}{2}S^2 C_{ss} \left( \sigma^2 \Delta t - \sqrt{\frac{2}{\pi}}k\sigma\sqrt{\Delta t} - \sigma^2 w^2 \Delta t - k\sigma|w|\sqrt{\Delta t} \right) \\ &\quad + O\left(\Delta t^{\frac{3}{2}}\right). \end{aligned}$$

Since the terms in the parentheses are independent of  $\mathcal{F}_{(j-1)\Delta t}$ , taking the conditional expectation with respect to  $\mathcal{F}_{(j-1)\Delta t}$ ,

$$E(\Delta H | \mathcal{F}_{(j-1)\Delta t}) = O\left(\Delta t^{\frac{3}{2}}\right)$$

which tends to zero as  $\Delta t \rightarrow 0$ . Note that  $S^2 C_{SS}$  is  $O\left(\Delta t^{\frac{1}{2}}\right)$  through the direct calculation for a call option. For options with general payoffs, we can obtain the derivative estimates for the option value with only the crude condition for the payoff (see Choi and Ku [5]). Then, we have

$$E(\Delta H^2) = O\left(\Delta t^2\right).$$

Therefore, following the law of large numbers (see for instance Feller [8]), Leland showed that the total hedging error over  $[0, T]$  tends to zero almost surely.

An option seller has a possibility of losing arbitrarily large amounts of money. If option sellers do not hedge frequently to reduce the risk, they may still face a significant loss. They have to chase the market and hedge their risk frequently until expiry to avoid serious damage.

In Figure 1-4, we assumed the time to expiry is 3 months and transaction costs are 4% and 1%, respectively. The standard deviation of the underlying asset is 20%, and the annualized interest rate is 10%. In Figure 1, we present results of 100 profit/loss simulations for the Black-Scholes-Leland option price when the hedging interval  $\Delta t$  is short, for example, 1 day. The right figure represents the histogram for the left

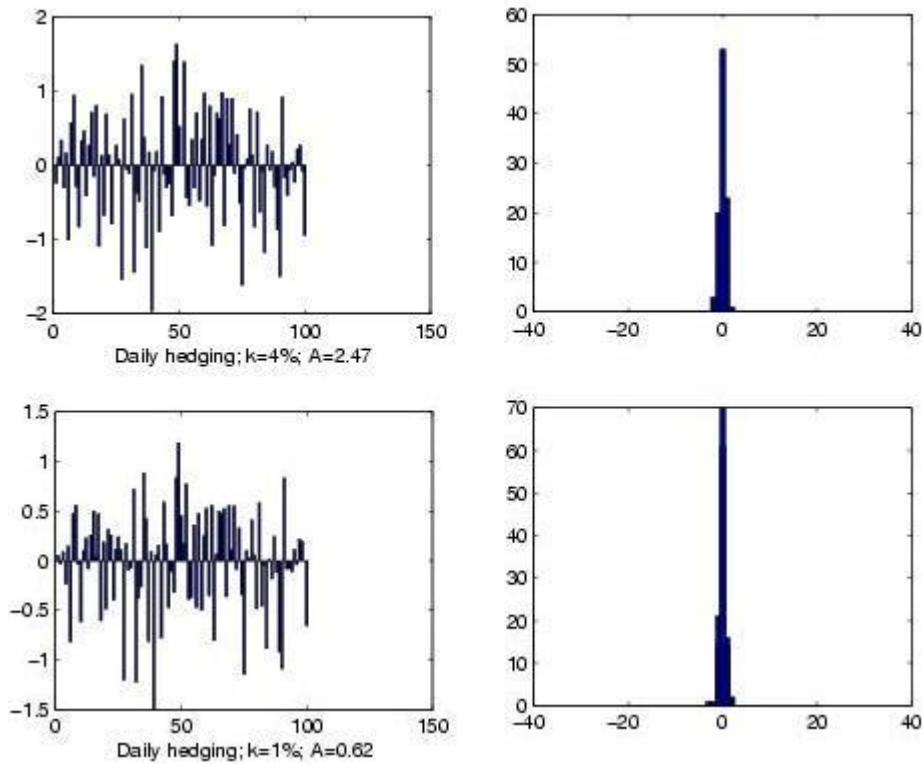


FIGURE 1. Results of profit/loss simulations for dynamic (daily) hedging of option sellers

one. Figure 2 shows how serious the risk of the option sellers is when the option is hedged just once.

### 3.2. Buyer's viewpoint

In this subsection, we consider option valuation and hedging problem for the long option position. The option buyer cannot replicate Leland's argument. The reason is that the long position of options which have positive gamma is very similar to the short position of options with negative gamma.

From the point of view of a seller who sells an option whose payoff function is convex, the option value  $C$  represents the hedging or replication costs. Therefore  $C$  should be increased by the premium charge on the transaction costs. On the other hand, from the point of view of an buyer who buys an option whose payoff function is convex, the option

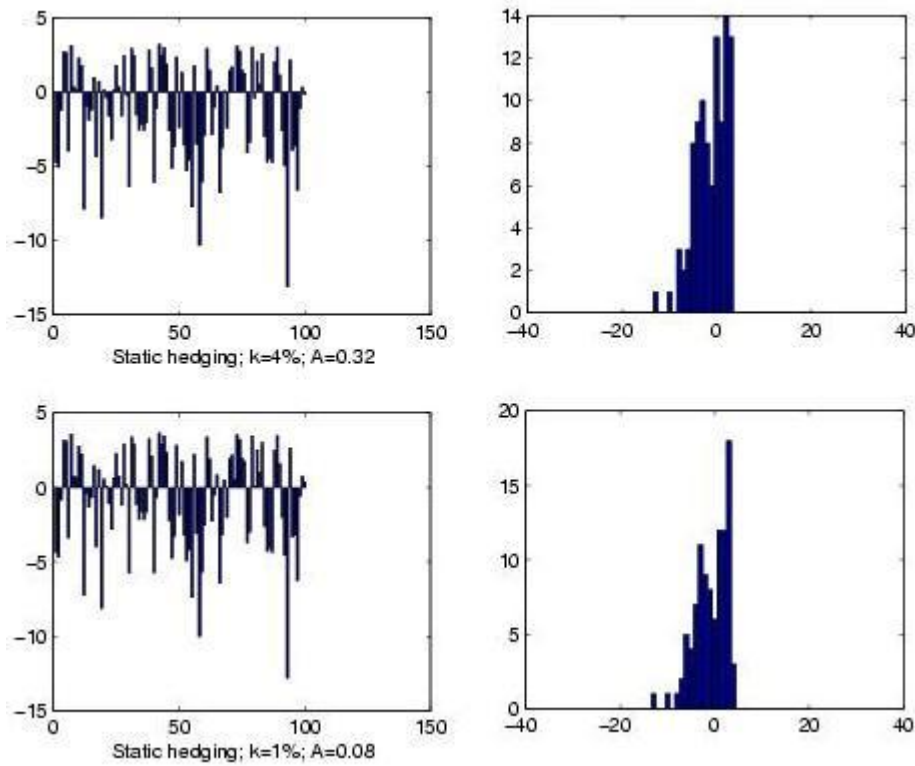


FIGURE 2. Results of profit/loss simulations for static hedging of option sellers

value  $C$  must be subtracted by that amount. Thus  $C$  satisfies the PDE

$$C_t + \frac{1}{2}\sigma_B^2 S^2 C_{SS} - rC + rSC_S = 0,$$

where  $\sigma_B$  is given by

$$\sigma_B = \sigma\sqrt{1-A}.$$

The above equation is same as the one derived from the Hoggard-Whalley-Wilmott nonlinear equation when the option gamma has negative sign, i.e., its payoff function is concave. Therefore its equation becomes an ill-posed problem when the Leland number  $A$  is bigger than one, so we cannot reduce the hedging interval  $\Delta t$  to zero. Hence we cannot apply the Leland's argument in this situation to make the hedging error arbitrary small.

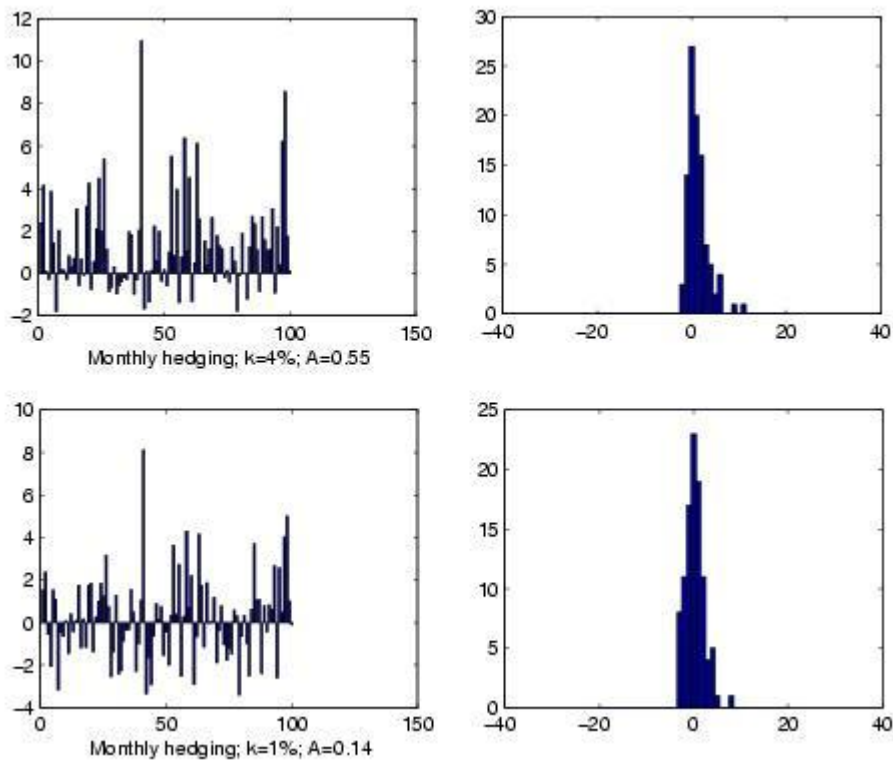


FIGURE 3. Results of profit/loss simulations for monthly hedging of option buyers

An option seller and an option buyer are very different. Normally, an option buyer goes into the market with the purpose of insurance or speculation. Thus we have to deal with the valuation and hedging problem of an option seller and of an option buyer differently.

One more thing is that an option buyer has a limited risk of loss. When an option buyer does not hedge frequently, he/she doesn't face a loss as much as an option seller. Even when an option buyer hedges every month or just once, the potential loss is not serious as we can see from the profit/loss simulation results in Figures 3 and 4. Here, the values of the parameters are depicted the same as in Figure 1 and 2.

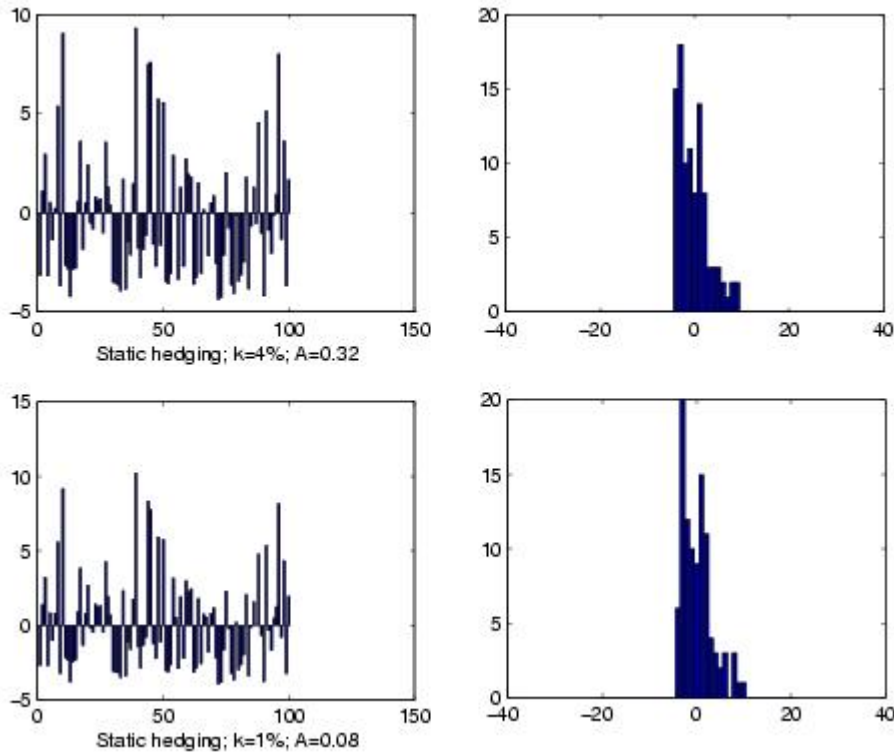


FIGURE 4. Results of profit/loss simulations for static hedging of option buyers

#### 4. Linear valuation and hedging method

In this section we show that we can decompose a short position of options with very general payoff into a short position and a long position of options each of which has the positive gamma.

The main idea is that we decompose the payoff function into two convex functions. By a theorem in real analysis, if the derivative of a payoff function is of bounded variation, it can be represented as a difference of two monotone increasing functions. Then we get, by integrating, the payoff function in the form of a difference of two convex functions. Since the convexity of the option value function is inherited from the convexity of the payoff function, two options obtained through the decomposition procedure have positive gamma.

As a typical example, it is well-known that short position of a vertical spread can be decomposed into one short call and another long call with different strike prices. But, we can apply this decomposition argument to almost all options with general payoffs.

**DEFINITION 4.1.** *Let  $\varphi : [0, \infty) \rightarrow \mathbb{R}$  be a function. We say  $\varphi$  satisfies the convex decomposition condition if it is a piecewise  $C^1$  function and its derivative is of bounded variation when restricted to any compact subinterval of  $[0, \infty)$ .*

The following is an easy consequence of the definition.

**PROPOSITION 4.1.** *Suppose  $\varphi$  is a function on  $[0, \infty)$  satisfying the convex decomposition condition. Then there exist convex functions  $\varphi_1$  and  $\varphi_2$  such that  $\varphi = \varphi_1 - \varphi_2$ .*

As alluded to above, this means that the seller's position of an option with a general payoff function is tantamount to holding a short position of one option with a convex payoff function and at the same time a long position of another option with a convex payoff function. Since when transaction costs are present, the economic positions of sellers and buyers of options are quite different, we treat each of these short and long position separately.

Now, consider the option price  $V$  for the payoff  $\varphi$  which is decomposed into  $\varphi = \varphi_1 - \varphi_2$ . The seller's price  $V^s$  can be computed by simply solving the Black-Scholes partial differential equation

$$V_t^s + \frac{1}{2}\sigma_A^2 S^2 V_{SS}^s - rV^s + rSV_S^s = 0$$

$$V^s(S, T) = \varphi_1(S)$$

where  $\sigma_A = \sigma\sqrt{1+A}$ . The buyer's price  $V^b(S, t)$  is also determined by solving the following linear equation

$$V_t^b + \frac{1}{2}\sigma_B^2 S^2 V_{SS}^b - rV^b + rSV_S^b = 0$$

$$V^b(S, T) = \varphi_2(S)$$

where  $\sigma_B = \sigma\sqrt{1-A}$  is the modified volatility to account for the buyer's transaction costs. The hedging interval  $\Delta t$  is chosen so that  $A$  is reasonably small. Once  $V^s(S, t)$  and  $V^b(S, t)$  are determined, the option price  $V(S, t) = V^s(S, t) - V^b(S, t)$  can be determined.

This means that the determination of  $V$  involves the solution of two partial differential equations, each of which can be independently obtained, and the hedging costs are already built in the volatility constants  $\sigma_A$  and  $\sigma_B$ .

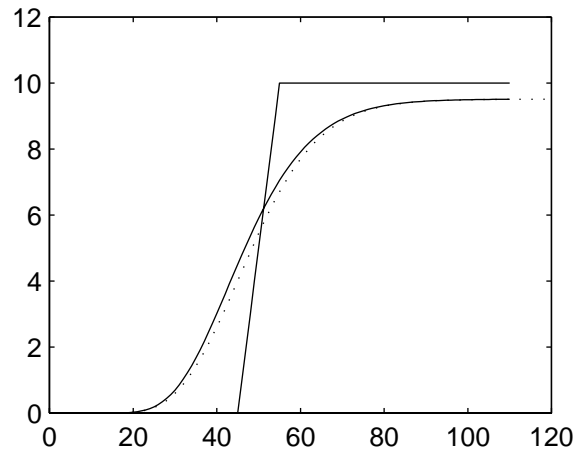


FIGURE 5. Comparison of option values through the Linear Valuation Method (the solid line) and the HWW method (the dotted line) for a vertical spread with strike price 45 and 55

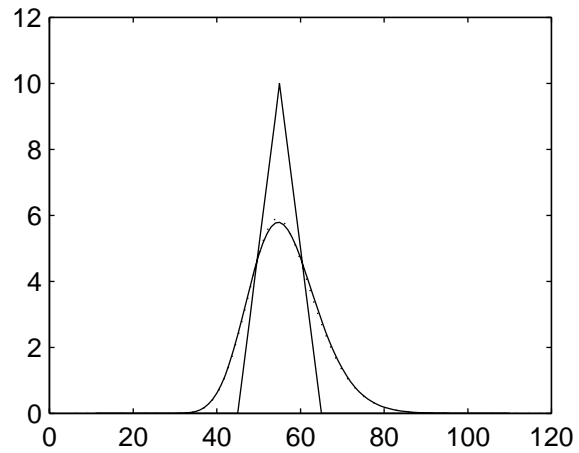


FIGURE 6. Comparison of option values through the Linear Valuation Method (the solid line) and the HWW method (the dotted line) for a butterfly spread with strike price 45, 55 and 65

To compare our linear valuation method with that of Hoggard et al. [10] when the Leland number  $A < 1$ , we take all parameter values the

same as in Hoggard et al. [10] and consider a vertical spread with strike price 45 and 55. The time to expiry is 6 months. Figure 5 shows the result of this computation. The solid line is the value function obtained through our method which is given by an explicit formula and the dotted line is the one from solving nonlinear partial differential equation in Hoggard et al. [10]. The other solid line represents the payoff function. Figure 6 shows the result for a butterfly spread with strike price 45, 55 and 65. The time to expiry is 1 month.

## 5. Modified valuation and hedging method

We have shown by the decomposition method of the previous section that a general payoff function can be decomposed into a difference of two convex functions. Then we found the option price by solving two linear partial differential equations separately. But the linear valuation and hedging method we presented in Section 4 might not perform so well, if the transaction costs are large and gamma changes rapidly.

In this section, we propose more efficient hedging scheme which costs less to hedge options. The decomposition of the option into a difference of two options is a fictitious scheme to extract the seller's and the buyer's position. However, in fact, there is only one option. Thus we can offset each hedging position, and hedge for the net position to get rid of unnecessary transaction costs.

First, the seller's price  $V^s$  and the buyer's price  $V^b$  are determined by solving the straightforward linear equation

$$V_t^s + \frac{1}{2}\sigma_A^2 S^2 V_{SS}^s - rV^s + rSV_S^s = 0$$

$$V^s(S, T) = \varphi_1(S),$$

and

$$V_t^b + \frac{1}{2}\sigma_B^2 S^2 V_{SS}^b - rV^b + rSV_S^b = 0$$

$$V^b(S, T) = \varphi_2(S),$$

respectively. Here,  $\sigma_A$  and  $\sigma_B$  are as in Section 4. Then, our modified hedging scheme becomes as follows. Consider the region  $\{(S, t) : V_{SS}^s(S, t) \geq V_{SS}^b(S, t)\}$ . In this region, the option price  $V = V^s - V^b$  has positive gamma and has the same kind of characteristics as the seller's price  $V^s$ . If we think  $V$  for the payoff  $\varphi = \varphi_1 - \varphi_2$ , we don't need to pay the unnecessarily transaction costs and only pay for the net change of the whole position. Thus, we can modify the option price  $V$  so that



reduce the transaction costs for both the seller's and buyer's position. In the same way as in Section 3.1, the hedging error over the hedging interval of which the length is  $\Delta t$  would be

$$\begin{aligned}\Delta H &= \Delta P - \Delta V - \text{Transaction Costs} \\ &= (V - V_S S)r\Delta t - V_t \Delta t - \frac{1}{2}V_{SS}(\Delta S)^2 - \frac{1}{2}kS^2 \left| V_{SS} \frac{\Delta S}{S} \right| \\ &\quad + O\left(\Delta t^{\frac{3}{2}}\right) \\ &= \left( rV - rSV_S - V_t - \frac{1}{2}S^2V_{SS} \left( \sigma^2 w^2 + \frac{k\sigma|w|}{\sqrt{\Delta t}} \right) \right) \Delta t \\ &\quad + O\left(\Delta t^{\frac{3}{2}}\right).\end{aligned}$$

To make the expected hedging error equal to zero, the option price  $V$  should satisfy

$$(5.1) \quad V_t + \frac{1}{2}\sigma_*^2 S^2 V_{SS} - rV + rSV_S = 0$$

where

$$\sigma_*^2 = \sigma^2 \left( 1 + \sqrt{\frac{2}{\pi}} \frac{k}{\sigma \sqrt{\Delta t}} \right).$$

On the other hand, considering the region  $\{(S, t) : V_{SS}^s(S, t) < V_{SS}^b(S, t)\}$ , the option price  $V = V^s - V^b$  has negative gamma and we can use the similar tactics as for the buyer's price  $V^b$ . Thus, we can choose the hedging interval for  $V$  to be  $l\Delta t$  for some positive number  $l$ , so that the corresponding Leland number is reasonably small and modify the option price  $V$  for  $\varphi = \varphi_1 - \varphi_2$ . Then, over the time interval of which the length is  $l\Delta t$ , a stock price  $S$  can be described by the following discrete equation

$$\frac{\Delta S}{S} = \mu l \Delta t + \sigma w \sqrt{l \Delta t}$$

where  $w$  is normally distributed random variable with mean zero and variance one.

Also, the hedging error inside the same time interval will be

$$\begin{aligned}
 \Delta H &= \Delta P - \Delta V - \text{Transaction Costs} \\
 &= (V - V_S S)rl\Delta t - V_t l\Delta t - \frac{1}{2}V_{SS}(\Delta S)^2 - \frac{1}{2}kS^2 \left| V_{SS} \frac{\Delta S}{S} \right| \\
 &\quad + O\left(\Delta t^{\frac{3}{2}}\right) \\
 &= \left( rV - rSV_S - V_t - \frac{1}{2}S^2V_{SS} \left( \sigma^2 w^2 - \frac{k\sigma|w|}{\sqrt{l\Delta t}} \right) \right) l\Delta t \\
 &\quad + O\left(\Delta t^{\frac{3}{2}}\right).
 \end{aligned}$$

To make the expected hedging error equal to zero, we have

$$(5.2) \quad V_t + \frac{1}{2}\sigma_*^2 S^2 V_{SS} - rV + rSV_S = 0$$

where

$$\sigma_*^2 = \sigma^2 \left( 1 - \sqrt{\frac{2}{\pi}} \frac{k}{\sigma \sqrt{l\Delta t}} \right).$$

Therefore, our modified valuation and hedging method now can be summarized as follows. We first find the seller's price  $V^s$  and buyer's price  $V^b$  as in Section 4, and then divide using the computed  $V^s$  and  $V^b$  into two regions. One is  $\{(S, t) : V_{SS}^s(S, t) \geq V_{SS}^b(S, t)\}$  and the other is  $\{(S, t) : V_{SS}^s(S, t) < V_{SS}^b(S, t)\}$ . Considering the whole position to get rid of unnecessary transaction costs, we control the hedging scheme according to the sign of gamma for the option price  $V^s - V^b$ . That is, we adopt the trading strategy which rehedges in  $\{(S, t) : V_{SS}^s(S, t) \geq V_{SS}^b(S, t)\}$  as often as  $V^s$  at each discrete time  $\Delta t$  and hedges in  $\{(S, t) : V_{SS}^s(S, t) < V_{SS}^b(S, t)\}$  as often as  $V^b$  once during  $l\Delta t$ . In this way, we finally obtain the modified option price  $V$  and hedging strategy for  $V$  by solving the partial differential equations (5.1) and (5.2) derived in each region.

To see the performance of our new method, we give an example: a vertical spread with strike price 45 and 55. We consider an underlying asset with annualized volatility 20%, interest rate of 10%, and the time to expiry of 3 months. The roundtrip transaction cost ratio is 2% and the hedging interval  $\Delta t = 1/240$  ( $\approx 1$  day). Then, the value of the Leland number is  $A = \sqrt{\frac{2}{\pi}} \frac{0.02}{0.2\sqrt{1/240}} = 1.24$ . Therefore, the method of Avellaneda and Parás works in this case. To use the method of Hoggard et al. [10], the hedging interval should be adjusted to make the Leland number  $A$  less than 1. Putting the hedging interval  $\Delta t = 1/48$  ( $\approx$

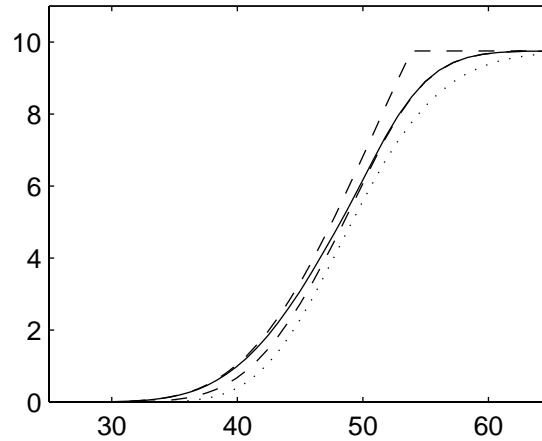


FIGURE 7. Comparison of option values through the Modified Valuation Method (the solid line), the AP Method (the upper dashed line), the HWW method (the lower dashed line), and with no transaction costs (the dotted line) for a vertical spread with strike price 45 and 55, the time to expiry is 3 months

1 week), the value of the Leland number becomes  $A = \sqrt{\frac{2}{\pi} \frac{0.02}{0.2\sqrt{1/48}}} = 0.55 < 1$ . However, our modified hedging method consists of two different rehedging schedules; we re hedge at every  $\Delta t = 1/240$  when gamma of the seller's part is bigger than that of the buyer's part, and re hedge at  $\Delta t = 1/48$  otherwise. By this hedging strategy, we obtain the modified option value. Figure 7 shows the result of this computation. The real line shows our modified value and the upper dashed line is the value of Avellaneda and Parás and the lower dashed line is that of Hoggard, Whalley, and Wilmott. We can see the real line is between two dashed lines. The dotted line represents the option value with no transaction costs.

In Figure 8, we give another computation result for a vertical spread with strike price 50 and 52, for which gamma changes very rapidly. We also simulated hedging this option using our hedging strategy. We assumed the initial price of the underlying asset was 50 and used two rehedging schedules,  $\Delta t = 1/240$  and  $\Delta t = 1/48$ , according to the value of gamma.

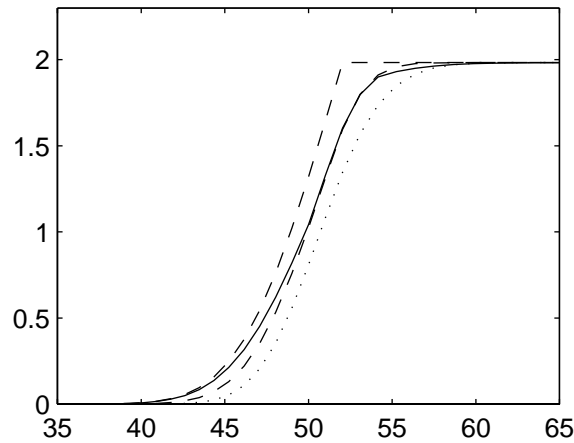


FIGURE 8. Comparison of option values through the Modified Valuation Method (the solid line), the AP Method (the upper dashed line), the HWW method (the lower dashed line), and with no transaction costs (the dotted line) for a vertical spread with strike price 50 and 52, the time to expiry is 1 month

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