OPTION PRICING AND HEDGING FOR THE STOCHASTIC VOLATILITY MODEL: NEW ITERATIVE PDE METHOD

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Abstract. We study the option valuation and hedging problem for the stochastic volatility model. We devise a new iterative partial differential equation method that yields a solution defined at discrete times, which converges to the true option value derived in Hull and White [4]. The advantages of our method are that it is well suited to the discrete time hedging procedure and is also amenable to numerical computation. Moreover, our method also allows us to deal with the valuation and hedging problem under transaction costs as in Leland [7].

1. Introduction

Black and Scholes [1] derived the partial differential equation

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf = 0$$

that must be satisfied by the option price $f = f(S,t)$, assuming that the stock price $S = S_t$ satisfies the stochastic differential equation

$$dS_t = \phi S_t dt + \sigma S_t dW_t, \tag{1.1}$$

where W_t is the standard Brownian motion, and ϕ, σ are constants. From this equation, they derived the Black-Scholes formula which is a basic formula in the derivative pricing. However, when the volatility σ is computed implicitly from the observed market prices of options, it is not in general constant. This implied volatility looks rather like U-shaped as a function of strike price which attains its minimum when strike price is near the current stock price. One usually calls this situation a "smile effect," or a "volatility smile."

This suggests that it is not very realistic to model the volatility as a constant. It is then natural to attempt to develop models in which the volatility is stochastic. One promising model is the one suggested by Hull and White [4]. Let σ be the volatility of the stock price

Key words and phrases. option pricing, hedging, stochastic volatility, iterative partial differential equations, replicating strategy, transaction costs.

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process S_t as in (1.1). They assumed that the variance $V = \sigma^2$ follows the independent lognormal diffusion process,

$$dV = \mu V dt + \xi V dZ_t \tag{1.2}$$

where μ is the local drift of V and ξ reflects the volatility of V. They, referring to Garman, induced and studied the following partial differential equation for the option price f

$$\frac{\partial f}{\partial t} + \frac{1}{2} V S^2 \frac{\partial^2 f}{\partial S^2} + \frac{1}{2} \xi^2 V^2 \frac{\partial^2 f}{\partial V^2} - rf + rS \frac{\partial f}{\partial S} + \mu V \frac{\partial f}{\partial V} = 0. \tag{1.3}$$

Let the mean variance $\overline{V}$ during the life of the option be given by $\overline{V} = \frac{1}{T} \int_0^T \sigma^2(t) dt$. They proved that the price of European option for their model can be obtained by integrating, over the distribution of $\overline{V}$, the Black-Scholes prices corresponding to each constant volatility $\sigma = \sqrt{\overline{V}}$. As a means of computation, they then derived a power series method, and gave a formula for the first few terms assuming that the drift μ of V is zero.

There are many other models dealing with the stochastic volatility. For example, Scott [8] proposed and analyzed a mean-reverting Ornstein-Uhlenbeck process for volatility, and showed that at least two options are necessary to form a hedging strategy and required two call options plus stock, the two call options must have different expiration dates to eliminate the uncertainty. He showed that the option prices can be computed via the Monte Carlo simulation method and used a Kalman filter model to estimate the current value of σ. Johnson and Shanno [5] assumed that the random term in the variance are completely diversifiable or there exists some asset that has the same random factor as the variance to derive the partial differential equation, and found the option price by the Monte Carlo simulation method. Stein and Stein [9] modeled stochastic volatility as the mean reverting process with a reflecting barrier at zero to prevent volatility becoming negative and developed an analytic approach based on Fourier inversion method. They all obtained partial differential equations with two state variables for the option price which are very similar to the one studied in [4], even though each has different assumption and approach.

In this paper, we take the Hull and White model and also assume that the stock price process follows (1.1), $V = \sigma^2$ satisfies the equation (1.2), and thus the European option price f satisfies the partial differential equation (1.3). Our approach, however, is different from that of Hull and White. Instead of studying the partial differential equation (1.3) directly, we devise an approximate problem based on certain iterative procedures at discrete time intervals. We then show that the solution of this iterative problem converges to the solution of the original problem (1.3). This method is naturally adapted to the easy numerical computation. The solution of this approximate iterative procedure is easy to handle. As a result, we obtain a discrete, hence more useful in practice, hedging method. Moreover, the same argument can be applied to obtain Leland's style option valuation and discrete hedging method in the presence of the transaction costs.

Let us now describe our method, which we call an iterative PDE family method, or in short, an iterative problem: Let T be the expiry of the option, and let 0 be the current time. Fix a small time interval $\Delta t = T/n$ for some positive integer n. This method is a way of defining a function $f^{(\Delta t)}(x, y, t)$ for $(x, y) \in \mathbb{R}^2_+$ and $t = T - k\Delta t$ for $k = 1, \cdots, n$.

Let us first describe how to define $f^{(\Delta t)}(x, y, T - \Delta t)$: For each fixed value of y at $t = T - \Delta t$, let σ be a constant over the time interval $[T - \Delta t, T]$ defined as $\sigma = \sqrt{y}$. Next, for each constant σ, define the following partial differential equation

$$L_\sigma^{(\Delta t)} u = \frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\xi^2 y^2 \frac{\partial^2 u}{\partial y^2} - ru + rx\frac{\partial u}{\partial x} + \mu y \frac{\partial u}{\partial y} = 0 \tag{1.4}$$

for all $(x, y) \in \mathbb{R}_+^2$ and $t \in [T-\Delta t, T]$. Let $f_\sigma^{(\Delta t)}(x, y, t)$ be the solution of the above partial differential equation (1.4) with the given initial condition $f_\sigma^{(\Delta t)}(x, y, T) = f(x, y, T)$. Then evaluate this $f_\sigma^{(\Delta t)}$ at $T - \Delta t$ for each $y = \sigma^2$, and set

$$f^{(\Delta t)}(x, y, T - \Delta t) = f_\sigma^{(\Delta t)}(x, y, T - \Delta t).$$

Note that the different values of y give rise to the different σ's. In order to define $f^{(\Delta t)}(x, y, T - \Delta t)$ for the different value of y, the different partial differential equation $L_\sigma^{(\Delta t)} f = 0$ should be used, hence the name "*family*." Repeating this way, we can recursively define $f^{(\Delta t)}(x, y, T - k\Delta t)$ from $f^{(\Delta t)}(x, y, T - (k-1)\Delta t)$ for $k = 1, \cdots, n$. Let us now formalize this procedure.

$<$ **Iterative PDE Family Method (in short, Iterative Problem)** $>$ (1.5)

Let $f(x, y, T)$ be the boundary condition, or equivalently, the contingent claim. Define $f^{(\Delta t)}(x, y, T - k\Delta t)$ for $(x, y) \in \mathbb{R}_+^2$ and $k = 1, 2, \cdots, n$, recursively as follows:

Step 1 (Initial Step) For each fixed $y > 0$ at $t = T - \Delta t$, let $\sigma = \sqrt{y}$. Define $f_\sigma^{(\Delta t)}(x, y, t)$ to be the solution of the initial value problem

$$\begin{cases} L_\sigma^{(\Delta t)} u = 0 \\ u(x, y, T) = f(x, y, T) \end{cases}$$

for $(x, y) \in \mathbb{R}_+^2$ and $t \in [T - \Delta t, T]$. Then set

$$f^{(\Delta t)}(x, y, T - \Delta t) = f_\sigma^{(\Delta t)}(x, y, T - \Delta t).$$

Step 2 (Recursive Step) Suppose $f^{(\Delta t)}(x, y, T - (k-1)\Delta t)$ is already found. Then for each fixed $y > 0$ at $t = T - k\Delta t$, let $\sigma = \sqrt{y}$. Define $f_\sigma^{(\Delta t)}(x, y, t)$ to be the solution of the problem

$$\begin{cases} L_\sigma^{(\Delta t)} u = 0 \\ u(x, y, T - (k-1)\Delta t) = f^{(\Delta t)}(x, y, T - (k-1)\Delta t) \end{cases}$$

for $(x, y) \in \mathbb{R}_+^2$ and $t \in [T - k\Delta t, T - (k-1)\Delta t]$. Then set

$$f^{(\Delta t)}(x, y, T - k\Delta t) = f_\sigma^{(\Delta t)}(x, y, T - k\Delta t).$$

Let $f^{(\Delta t)}(x,y,t)$ be the solution of the Iterative Problem (1.5). Then $f^{(\Delta t)}(x,y,t)$ is initially defined only for $t = T - k\Delta t$ for $k = 0, 1, \cdots, n$. In fact, we will only need the value of $f^{(\Delta t)}(x,y,t)$ for $t = T - k\Delta t$, for $k = 0, 1, \cdots, n$ except that we need to be able to make sense out of $\frac{\partial f^{(\Delta t)}}{\partial t}(x, y, T - k\Delta t)$, in which case $\frac{\partial f^{(\Delta t)}}{\partial t}(x, y, T - k\Delta t)$ simply means $\frac{\partial f_\sigma^{(\Delta t)}}{\partial t}(x, y, T - k\Delta t)$ where $\sigma = \sqrt{y}$.

When one notes that the variable y is the square of the volatility constant σ in Equation (1.1), our method looks very natural. There are many ways of fixing y as a constant in Equation (1.3) to arrive at something like $L_\sigma^{(\Delta t)}$. But we only change y in the coefficient of the second partial derivative with respect to x in Equation (1.3), while leaving other y in the coefficient of the first partial derivative with respect to y in Equation (1.3). The reason for this is that this replacement makes it easy to transform $L_\sigma^{(\Delta t)} = 0$ to the two dimensional standard heat equation by suitable change of variables. Once the equation is rewritten in the standard heat equation form, the solution can be explicitly written down by integrating with the two dimensional heat kernel.

In Section 2, we justify our approach. Theorem 2.2, which forms the basis of our work, shows that the function constructed by the Iterative Problem (1.5) converges to the solution of Equation (1.3) with the same initial condition as $\Delta t \to 0$. This fact says that we can use $f^{(\Delta t)}$ as an approximate value of the option price. Note that the numeric value given in Hull and White [4] is also an approximate one, since they only use first few terms in their power series expansion.

The rest of this paper is devoted to finding the hedging method. In order to use any option valuation method in practice, one has to figure out a suitable hedging strategy. Although Hull and White did not deal with this problem, we take up this issue in Section 4 and 5. In Section 4, we present a discrete time replicating strategy by using $f^{(\Delta t)}$ defined by the Iterative PDE family Method with a hedging interval Δt. And we prove that the hedging error goes to zero as $\Delta t \to 0$. This kind of discrete time hedging, in fact, is more realistic in actual trading securities than the continuous one. In Section 5, we consider the valuation and hedging problem with transaction costs from the option seller's view-point. Namely, as Leland [7] did, we calculate the premium associated with having to pay the transaction costs. The arguments are similar to those in Section 4.

2. Convergence of the Solution of the Iterative PDE Family to the Solution of the Hull-White Equation

In this section, we prove that $f^{(\Delta t)}(x,y,t)$ defined by the Iterative Problem (1.5) converges to the solution of the Hull-White equation as $\Delta t \to 0$. If no confusion is possible, we simply drop Δt from $f^{(\Delta t)}$ and $L^{(\Delta t)}$.

Let us first set up some notations, and quick review on some relevant facts in Hull and White [4]. First, the asset price process S and its variance $V = \sigma^2$ are assumed to obey the following stochastic differential equations

$$dS = \phi S dt + \sigma S dW_t$$
$$dV = \mu V dt + \xi V dZ_t$$

where (W_t, Z_t) is a standard two-dimensional Brownian motion and ϕ, μ and ξ are constants. They showed that the option price $\tilde{f}$ is given by the following formula

$$\tilde{f}(x, \sigma_t^2, t) = \int C(\overline{V})h(\overline{V}|\sigma_t^2)d\overline{V} \tag{2.1}$$

where $\overline{V}$ is the mean variance over the life of the derivative security, $h(\overline{V}|\sigma_t^2)$ is the conditional density function of $\overline{V}$ given the variance at time t, and $C(\overline{V})$ is the Black-Scholes price with the volatility $\overline{V}$. They also proved that $\tilde{f}(x, y, t)$ is the solution of the following problem

$$\begin{cases} \frac{\partial \tilde{f}}{\partial t} + \frac{1}{2}yx^2\frac{\partial^2 \tilde{f}}{\partial x^2} + \frac{1}{2}\xi^2 y^2 \frac{\partial^2 \tilde{f}}{\partial y^2} - r\tilde{f} + rx\frac{\partial \tilde{f}}{\partial x} + \mu y \frac{\partial \tilde{f}}{\partial y} = 0 \\ \tilde{f}(x, y, T) = (x - K)^+. \end{cases} \tag{2.2}$$

First, we show that the term $yx^2\frac{\partial^2 \tilde{f}}{\partial x^2}$ for the solution of (2.2) is uniformly bounded. Since PDE (2.2) is difficult to handle directly, we give a probabilistic proof. This is perhaps a good place to clarify the nature of various estimates we derived in this paper. First of all, one should note that $\frac{\partial^2 \tilde{f}}{\partial x^2}$ blows up as $t \to T$. This is due to the discontinuity of the first derivative of $f(x, y, T) = (x - K)^+$ at $x = K$. This causes a trouble to most of the uniform estimates. This fact is often overlooked in the literatures. One way of getting around this difficulty is to tame the corner of the payoff function in a C^∞ manner in a small neighborhood of the corner. Then all estimates would work as intended. Another approach is to back track the time interval to $[0, T - \delta]$ instead of $[0, T]$ and use the value $\tilde{f}(x, y, T - \delta)$ as the boundary condition. Then all derivative estimates and hedging error estimates would work up to time $T - \delta$. But, since δ is arbitrary, this slight modification does not cause any serious trouble. Therefore, from now on, all estimates are obtained assuming that the boundary condition meets this modification requirement.

Lemma 2.1. *Let $\tilde{f}(x, y, t)$ be the price of European call option with stochastic volatility that satisfies* (2.2)*. Then there exists a constant C independent of x, y and t such that*

$$yx^2\frac{\partial^2 \tilde{f}}{\partial x^2}(x, y, t) \le C.$$

Proof. From the formula (2.1),

$$\begin{aligned} yx^2\frac{\partial^2 \tilde{f}}{\partial x^2}(x, y, t) &= \int yx^2\frac{\partial^2 C}{\partial x^2}(\overline{V})h(\overline{V}|y = \sigma_t^2)d\overline{V} \\ &= \int \overline{V}x^2\frac{\partial^2 C}{\partial x^2}(\overline{V})\frac{y}{\overline{V}}h(\overline{V}|y = \sigma_t^2)d\overline{V}. \end{aligned}$$

It is easy to see that the original Black-Scholes formula can be used to show that $\overline{V}x^2\frac{\partial^2 C}{\partial x^2}(\overline{V})$ is bounded by some constant M independent of x and $\overline{V}$. (In fact, M depends only on

$T-t$, and it blows up $t \to T$. See the comment preceding Lemma 2.1.) Therefore the last term is bounded above by

$$M\int_{\sigma_t^2}^{\infty} \frac{y}{y'} h(\overline{V}=y'|y=\sigma_t^2)dy' + M\int_0^{\sigma_t^2} \frac{y}{y'} h(\overline{V}=y'|\sigma_t^2)dy'$$
$$\leq M\left\{\sum_{n=1}^{\infty}\frac{1}{n}P\big[n\sigma_t^2 \leq \overline{V} \leq (n+1)\sigma_t^2|\sigma_t^2\big] + \sum_{m=1}^{\infty}(m+1)P\big[\frac{\sigma_t^2}{m+1} \leq \overline{V} \leq \frac{\sigma_t^2}{m}|\sigma_t^2\big]\right\}.$$

The first term in the above parenthesis is less than

$$\sum_{n=1}^{\infty}\frac{1}{n}P\big[\overline{V} \geq n\sigma_t^2|\sigma_t^2\big] \leq \sum_{n=1}^{\infty}\frac{1}{n}P\big[\max_{t\leq\tau\leq T}\sigma_\tau^2 \geq n\sigma_t^2|\sigma_t^2\big].$$

Using the formula $\sigma_\tau^2(\omega) = e^{(\mu-\xi^2/2)(\tau-t)+\xi Z_{\tau-t}(\omega)}\sigma_t^2$ for $\tau > t$, and considering the distribution of the running maximum of Brownian motion [6, p.96], we get

$$\sum_{n=1}^{\infty}\frac{1}{n}P\big[\max_{0\leq s\leq T-t} Z_s \geq \frac{1}{\xi}\log n - |(\frac{\mu}{\xi}-\frac{\xi}{2})|(T-t)\big]$$
$$\leq \int_1^{\infty}\frac{1}{y'}\int_{\frac{1}{\xi}\log y' - |(\frac{\mu}{\xi}-\frac{\xi}{2})|(T-t)}^{\infty} \frac{2}{\sqrt{2\pi(T-t)}}e^{-z^2/2(T-t)}dz,$$

which converges. Finally, we can show the second term is also bounded by means of the similar computation with the running minimum of Brownian motion. ∎

We now prove a theorem which is the foundation of our subsequent investigation, so that $f(x,y,t)$ is used as a proxy for $\tilde{f}(x,y,t)$ in devising the hedging method.

Theorem 2.2. *Let $\tilde{f}(x,y,t)$ be the solution of* (2.2) *and $f(x,y,t)$ be the solution of the Iterative Problem* (1.5). *Then, at each $t < T$, $f(x,y,t)$ converges to $\tilde{f}(x,y,t)$ as $\Delta t \to 0$.*

Proof. Let σ be a constant to be chosen appropriately for each iteration step below. Define the following change of variables

$$u = \frac{\sqrt{2}}{\sigma}\log x/K + \left(-\frac{\sqrt{2}}{\sigma}r + \frac{\sigma}{\sqrt{2}}\right)(t-T),$$
$$v = \frac{\sqrt{2}}{\xi}\log y/\sigma^2 + \left(-\frac{\sqrt{2}}{\xi}\mu + \frac{\xi}{\sqrt{2}}\right)(t-T),$$
$$s = -(t-T),$$

for $x, y \in \mathbb{R}_+$ and $t \in [T-\Delta t, T]$. Define $g(u,v,s)$ for $u, v \in \mathbb{R}, s \in [0, \Delta t]$ by

$$g(u,v,s) = e^{-r(t-T)}f_\sigma(x,y,t).$$

Since $L_\sigma f_\sigma = 0$, this change of variables formula makes $g(u,v,s)$ satisfy the standard heat equation, *i.e.*, $\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} - \frac{\partial g}{\partial s} = 0$. From the equation

$$L_\sigma(f_\sigma - \tilde{f}) = \frac{1}{2}(y-\sigma^2)x^2\frac{\partial^2 \tilde{f}}{\partial x^2},$$

the difference $\varepsilon(u,v,s) = e^{-r(t-T)}(f_\sigma - \tilde{f})(x,y,t)$ is a solution of the following inhomogeneous heat equation

$$\frac{\partial^2 \varepsilon}{\partial u^2} + \frac{\partial^2 \varepsilon}{\partial v^2} - \frac{\partial \varepsilon}{\partial s} = h(u,v,s) \equiv \frac{1}{2}(y-\sigma^2)x^2\frac{\partial^2 \tilde{f}}{\partial x^2}$$
$$\varepsilon(u,v,0) = 0.$$

By the basic theorem of parabolic PDE, see for instance [3], we can represent

$$\varepsilon(u,v,s) = \int_0^s \int_{\mathbb{R}^2} \frac{1}{4\pi\tau} e^{-(z_1^2+z_2^2)/4\tau} h(u-z_1, v-z_2, s-\tau)dz_1dz_2d\tau.$$

To estimate $\varepsilon(u,v,s)$, we need to bound $h(u,v,s)$. Note that $\frac{y}{\sigma^2} = e^{(\alpha v + \beta s)}$ where $\alpha = \frac{\xi}{\sqrt{2}} > 0$ and $\beta = -\mu + \frac{\xi^2}{2}$. Then, we can write $y - \sigma^2 = y\left(1 - e^{-\alpha v - \beta s}\right)$. Using Lemma 2.1,

$$|h(u,v,s)| = |\frac{1}{2}(y-\sigma^2)x^2\frac{\partial^2 \tilde{f}}{\partial x^2}|$$
$$\leq C|1 - e^{-\alpha v - \beta s}|$$

for some constant C. At $t = T - \Delta t$, the values of x, y and t correspond to

$$\overline{u} = \frac{\sqrt{2}}{\sigma}\log x/K - \left(-\frac{\sqrt{2}}{\sigma}r + \frac{\sigma}{\sqrt{2}}\right)\Delta t,$$
$$\overline{v} = -\left(-\frac{\sqrt{2}}{\xi}\mu + \frac{\xi}{\sqrt{2}}\right)\Delta t,$$
$$\overline{s} = \Delta t.$$

Therefore, we have

$$|\,\varepsilon(\overline{u},\overline{v},\overline{s})| \leq \int_0^{\Delta t}\int_{\mathbb{R}^2} \frac{1}{4\pi\tau}e^{-(z_1^2+z_2^2)/4\tau}|h(\overline{u}-z_1, \overline{v}-z_2, \overline{s}-\tau)|\; dz_1dz_2d\tau$$
$$\leq C\int_0^{\Delta t}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\tau}}e^{-z_2^2/4\tau}|e^{\alpha z_2}e^{\beta\tau} - 1|\; dz_2d\tau.$$

When $|z_2| \leq \delta$ for some $\delta > 0$, the power series expansion of $e^{\alpha z_2}$ gives

$$|e^{\alpha z_2}e^{\beta\tau} - 1| \leq |e^{\beta\tau} - 1| + e^{\alpha\delta}e^{\beta\tau}\alpha|z_2|$$
$$\leq |e^{\beta\tau} - 1| + \gamma|z_2|$$

for some constant $\gamma > 0$. Therefore, we have

$$
\begin{aligned}
|\,\varepsilon(\overline{u},\overline{v},\overline{s})| &\le C\Big\{\int_0^{\Delta t}\int_{-\delta}^{\delta}\frac{1}{\sqrt{4\pi\tau}}e^{-z_2^2/4\tau}|e^{\alpha z_2}e^{\beta\tau}-1|\;dz_2d\tau \\
&\quad+\int_0^{\Delta t}\int_{\delta}^{\infty}\frac{1}{\sqrt{4\pi\tau}}e^{-z_2^2/4\tau}e^{\alpha z_2}\;dz_2d\tau+\int_0^{\Delta t}\int_{-\infty}^{-\delta}\frac{1}{\sqrt{4\pi\tau}}e^{-z_2^2/4\tau}\;dz_2d\tau\Big\} \\
&\le C\Big\{\int_0^{\Delta t}|e^{\beta\tau}-1|\;d\tau+2\gamma\int_0^{\Delta t}\int_0^{\delta}\frac{1}{\sqrt{4\pi\tau}}e^{-z_2^2/4\tau}z_2\;dz_2d\tau \\
&\quad+\int_0^{\Delta t}\int_{\delta}^{\infty}\frac{1}{\sqrt{4\pi\tau}}e^{-(z_2-2\alpha\tau)^2/4\tau}e^{\alpha^2\tau}\;dz_2d\tau+\int_0^{\Delta t}\int_{-\infty}^{-\delta}\frac{1}{\sqrt{4\pi\tau}}e^{-z_2^2/4\tau}\;dz_2d\tau\Big\}.
\end{aligned}
$$

The second integral in the right-hand side is equal to

$$\frac{2\gamma}{\sqrt{\pi}}\int_0^{\Delta t}\sqrt{\tau}(1-e^{-\delta^2/4\tau})\;d\tau$$

and the rest of terms decay very fast when Δt gets small. Hence, we get $\big|(f_\sigma-\tilde{f})(x,\sigma^2,T-\Delta t)\big|\le C(\Delta t)^{3/2}$ for some new constant C and small Δt. Since $\sigma=\sqrt{y}$ is arbitrary and $f(x,y,T-\Delta t)=f_{\sqrt{y}}(x,y,T-\Delta t)$, we have

$$\big|(f-\tilde{f})(x,y,T-\Delta t)\big|\le C(\Delta t)^{3/2}.$$

To compare f with $\tilde{f}$ at $t=T-2\Delta t$ in the next step, we introduce a function φ such that $L_\sigma\varphi=0$ for $(x,y,t)\in\mathbb{R}^+\times\mathbb{R}^+\times[T-2\Delta t,T-\Delta t]$ with $\varphi(x,y,T-\Delta t)=\tilde{f}(x,y,T-\Delta t)$. Then, by the argument in the previous step, we can easily show that $\big|(\varphi-\tilde{f})(x,y,T-2\Delta t)\big|\le C(\Delta t)^{3/2}$. To bound $(f-\varphi)(x,y,T-2\Delta t)$, let us first fix σ and consider $(f_\sigma-\varphi)(x,y,t)$ for $t\in[T-2\Delta t,T-\Delta t]$. Then, $f_\sigma-\varphi$ satisfies the homogeneous equation $L_\sigma(f_\sigma-\varphi)=0$ with the initial condition $\big|(f_\sigma-\varphi)(x,y,T-\Delta t)\big|\le C(\Delta t)^{3/2}$. This implies $\big|(f_\sigma-\varphi)(x,y,T-2\Delta t)\big|\le Ce^{-r\Delta t}(\Delta t)^{3/2}$. Therefore, we have $\big|f-\varphi\big|(x,y,T-2\Delta t)\le C(\Delta t)^{3/2}$, since σ is arbitrary and $f(x,y,T-2\Delta t)=f_{\sqrt{y}}(x,y,T-2\Delta t)$. Thus combining the bounds for $f-\varphi$ and $\varphi-\tilde{f}$, we have

$$\big|f-\tilde{f}\big|(x,y,T-2\Delta t)\le 2C(\Delta t)^{3/2}$$

for some constant C. Repeating the same argument, therefore, $f(x,y,t)$ converges to $\tilde{f}(x,y,t)$ as $\Delta t\to 0$. Hence the proof is complete. ∎

3. Derivative Estimates for the Solution of the Iterative Problem

In this section, we obtain the derivative estimates of the solution $f(x,y,t)$ of the Iterative Problem (1.5). These derivative estimates are important ingredients in computing the hedging error in Section 4 and 5. The advantage of using the operator L_σ is that this operator can be transformed to the standard two dimensional heat operator via change of variables.

Lemma 3.1. *Suppose that f is the solution of the Iterative Problem* (1.5). *Then $yx^2\frac{\partial^2 f}{\partial x^2}$ (x,y,t) is uniformly bounded for $t = T - k\Delta t$ $(k = 1, \cdots, n)$ by a constant C independent of x, y and Δt.*

Proof. Fix $\sigma > 0$ and let $L_\sigma f_\sigma = 0$. Applying the operator $yx^2\frac{\partial^2}{\partial x^2}$, we have

$$yx^2\frac{\partial^2}{\partial x^2}L_\sigma f_\sigma = 0.$$

Upon rewriting the above equation, we get a new equation

$$M_\sigma\psi \equiv \frac{\partial}{\partial t}\psi + \frac{1}{2}\sigma^2x^2\frac{\partial^2}{\partial x^2}\psi + \frac{1}{2}\xi^2y^2\frac{\partial^2}{\partial y^2}\psi + (\xi^2 - \mu - r)\psi + rx\frac{\partial}{\partial x}\psi + (\mu - \xi^2)y\frac{\partial}{\partial y}\psi = 0$$

where $\psi = yx^2\frac{\partial^2 f_\sigma}{\partial x^2}$. The operator M_σ is essentially the same type of operator as L_σ with having a bit changed coefficients. Thus one can use the maximum principle argument for ψ via the change of variables as in Theorem 2.2.

For $x, y \in \mathbb{R}_+$ and $t \in [T - \Delta t, T]$, define

$$u = \frac{\sqrt{2}}{\sigma}\log x/K + \left(-\frac{\sqrt{2}}{\sigma}\overline{r} + \frac{\sigma}{\sqrt{2}}\right)(t - T),$$
$$v = \frac{\sqrt{2}}{\xi}\log y/\sigma^2 + \left(-\frac{\sqrt{2}}{\xi}\overline{\mu} + \frac{\xi}{\sqrt{2}}\right)(t - T),$$
$$s = -(t - T),$$

where $\overline{r} = r + \mu - \xi^2$ and $\overline{\mu} = \mu - \xi^2$. With respect to these new variables, define a new function $g(u, v, s)$ by

$$g(u, v, s) = e^{-\overline{r}(t-T)}yx^2\frac{\partial^2 f_\sigma}{\partial x^2}(x, y, t).$$

It is easy to see that g satisfies the standard heat equation $\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} - \frac{\partial g}{\partial s} = 0$. The initial value of f_σ is $f(x, y, T)$ where $f(x, y, T) = \tilde{f}(x, y, T)$. Thus $|yx^2\frac{\partial^2 f_\sigma}{\partial x^2}(x, y, T)| = |yx^2\frac{\partial^2 f}{\partial x^2}(x, y, T)| \le C$ by Lemma 2.1, *i.e.*, $|g(u, v, 0)| \le C$. Then the maximum principle says that $|g(u, v, \Delta t)| \le C$. Thus $|yx^2\frac{\partial^2 f_\sigma}{\partial x^2}(x, y, T - \Delta t)| \le Ce^{-\overline{r}\Delta t}$. Since σ is arbitrary and $f(x, y, T - \Delta t) = f_{\sqrt{y}}(x, y, T - \Delta t)$, letting $\sigma = \sqrt{y}$ leads to

$$|yx^2\frac{\partial^2 f}{\partial x^2}(x, y, T - \Delta t)| \le Ce^{-\overline{r}\Delta t}.$$

Then we can repeat the above argument by introducing new change of variables (u, v, s) for (x, y, t) and $t \in [T - 2\Delta t, T - \Delta t]$, where $t - T$ above is replaced by $t - (T - \Delta t)$. Namely, for any $\sigma > 0$, $M_\sigma(yx^2\frac{\partial^2 f_\sigma}{\partial x^2}) = 0$ for $t \in [T - 2\Delta t, T - \Delta t]$ with the boundary condition $yx^2\frac{\partial^2 f_\sigma}{\partial x^2}(x, y, T - \Delta t) = yx^2\frac{\partial^2 f}{\partial x^2}(x, y, T - \Delta t)$, which is shown to be bounded above by $Ce^{-\overline{r}\Delta t}$. Repeating the argument by the change of variables, we can conclude that

$$|yx^2\frac{\partial^2 f}{\partial x^2}(x,y,T-2\Delta t)| \le Ce^{-\overline{r}2\Delta t}.$$

Therefore, we can easily see that

$$|yx^2\frac{\partial^2 f}{\partial x^2}(x,y,t)| \le Ce^{-\overline{r}(T-t)}. \quad \blacksquare$$

Lemma 3.2. *Suppose that $f(x,y,t)$ is the solution of the Iterative Problem* (1.5). *Then, at each $t = T - k\Delta t$ $(k = 1, \cdots, n)$, $|y\frac{\partial f}{\partial y}(x,y,t)| \le C(\Delta t)^{\frac{1}{2}}$ for some constant C independent of x, y and Δt.*

Proof. By definition of the solution of the Iterative Problem (1.5),

$$\begin{aligned} y\frac{\partial f}{\partial y}(x,y,T-k\Delta t)|_{y=\sigma^2} &= \lim_{\widehat{\sigma}^2\to\sigma^2} \sigma^2 \frac{f(x,\widehat{\sigma}^2,T-k\Delta t) - f(x,\sigma^2,T-k\Delta t)}{\widehat{\sigma}^2-\sigma^2} \\ &= \lim_{\widehat{\sigma}^2\to\sigma^2} \sigma^2 \frac{f_{\widehat{\sigma}}(x,\widehat{\sigma}^2,T-k\Delta t) - f_\sigma(x,\sigma^2,T-k\Delta t)}{\widehat{\sigma}^2-\sigma^2}, \end{aligned}$$

where $f_{\widehat{\sigma}}(x,y,t)$ and $f_\sigma(x,y,t)$ are functions defined for $x, y \in \mathbb{R}_+$ and $t \in [T-k\Delta t, T-(k-1)\Delta t]$ satisfying the partial differential equations

$$\begin{aligned} L_{\widehat{\sigma}} f_{\widehat{\sigma}} &= \frac{\partial f_{\widehat{\sigma}}}{\partial t} + \frac{1}{2}\widehat{\sigma}^2x^2\frac{\partial^2 f_{\widehat{\sigma}}}{\partial x^2} + \frac{1}{2}\xi y^2\frac{\partial^2 f_{\widehat{\sigma}}}{\partial y^2} - rf_{\widehat{\sigma}} + rx\frac{\partial f_{\widehat{\sigma}}}{\partial x} + \mu y\frac{\partial f_{\widehat{\sigma}}}{\partial y} = 0, \\ L_\sigma f_\sigma &= \frac{\partial f_\sigma}{\partial t} + \frac{1}{2}\sigma^2x^2\frac{\partial^2 f_\sigma}{\partial x^2} + \frac{1}{2}\xi^2 y^2\frac{\partial^2 f_\sigma}{\partial y^2} - rf_\sigma + rx\frac{\partial f_\sigma}{\partial x} + \mu y\frac{\partial f_\sigma}{\partial y} = 0 \end{aligned}$$

with the same initial condition $f_{\widehat{\sigma}}(x,y,T-(k-1)\Delta t) = f_\sigma(x,y,T-(k-1)\Delta t) = f(x,y,T-(k-1)\Delta t)$. Rewriting the second one, we have

$$L_{\widehat{\sigma}} f_\sigma = \frac{\partial f_\sigma}{\partial t} + \frac{1}{2}\widehat{\sigma}^2x^2\frac{\partial^2 f_\sigma}{\partial x^2} + \frac{1}{2}\xi^2 y^2\frac{\partial^2 f_\sigma}{\partial y^2} - rf_\sigma + rx\frac{\partial f_\sigma}{\partial x} + \mu y\frac{\partial f_\sigma}{\partial y} = \frac{1}{2}(\widehat{\sigma}^2-\sigma^2)x^2\frac{\partial^2 f_\sigma}{\partial x^2}.$$

Thus,

$$L_{\widehat{\sigma}}\left(\frac{f_{\widehat{\sigma}} - f_\sigma}{\widehat{\sigma}^2-\sigma^2}\right) = -\frac{1}{2}x^2\frac{\partial^2 f_\sigma}{\partial x^2} \tag{3.1}$$

holds for all $x, y \in \mathbb{R}_+$ and $t \in [T-k\Delta t, T-(k-1)\Delta t]$.

Let us first give a proof for the case $k = 1$. Let us use the change of variables given at the beginning of the proof of Theorem 2.2. Note that a slight modification of the argument in the proof of Lemma 3.1 shows that $yx^2\frac{\partial^2 f_\sigma}{\partial x^2}(x,y,t)$ is bounded. Thus at any $x, y \in \mathbb{R}_+$ and $t \in [T-\Delta t, T]$, we have

$$\begin{aligned} |\sigma^2 x^2\frac{\partial^2 f_\sigma}{\partial x^2}| &\le |\frac{\sigma^2}{y}yx^2\frac{\partial^2 f_\sigma}{\partial x^2}| \\ &\le Ce^{-\left(\frac{\xi}{\sqrt{2}}v + (-\mu+\frac{\xi^2}{2})s\right)}. \end{aligned} \tag{3.2}$$

Now we evaluate $y\frac{\partial f}{\partial y}$ at $x, y = \sigma^2$ and $t = T - \Delta t$. Using the same change of variables formula over $t \in [T - \Delta t, T]$ as in Theorem 2.2, we write $\overline{u}, \overline{v}, \overline{s}$ as the values of u, v, s, respectively at $t = T - \Delta t$. Then, since $f_{\widehat{\sigma}}(x, y, T) = f_\sigma(x, y, T)$, (3.1) and (3.2) imply that

$$\begin{aligned}|\sigma^2 \frac{f_{\widehat{\sigma}} - f_\sigma}{\widehat{\sigma}^2 - \sigma^2}(x, \sigma^2, T - \Delta t)| &\leq \int_0^{\Delta t} \int_{\mathbb{R}^2} \frac{1}{4\pi\tau} e^{-(z_1^2+z_2^2)/4\tau} \left| \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f_\sigma}{\partial x^2} \right| (\overline{u} - z_1, \overline{v} - z_2, \overline{s} - \tau) dz_1 dz_2 d\tau \\ &\leq C \int_0^{\Delta t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\tau}} e^{-z_2^2/4\tau} e^{-\frac{\xi}{\sqrt{2}}(\overline{v} - z_2)} e^{-(-\mu + \frac{\xi^2}{2})(\Delta t - \tau)} dz_2 d\tau \\ &= C \int_0^{\Delta t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\tau}} e^{-z_2^2/4\tau} e^{\alpha z_2} e^{\beta\tau}\, dz_2 d\tau\end{aligned}$$

where $\alpha = \frac{\xi}{\sqrt{2}}$ and $\beta = -\mu + \frac{\xi^2}{2}$. On the other hand, since

$$\frac{f_{\widehat{\sigma}}(x, \widehat{\sigma}^2, t) - f_\sigma(x, \sigma^2, t)}{\widehat{\sigma}^2 - \sigma^2} = \frac{f_{\widehat{\sigma}}(x, \widehat{\sigma}^2, t) - f_{\widehat{\sigma}}(x, \sigma^2, t)}{\widehat{\sigma}^2 - \sigma^2} + \frac{f_{\widehat{\sigma}}(x, \sigma^2, t) - f_\sigma(x, \sigma^2, t)}{\widehat{\sigma}^2 - \sigma^2},$$

letting $\widehat{\sigma}^2 \to \sigma^2$ gives

$$\begin{aligned}y\frac{\partial f}{\partial y}(x, \sigma^2, T - \Delta t) =& e^{-r\Delta t} \int_{\mathbb{R}^2} \frac{1}{4\pi\tau} e^{-(z_1^2+z_2^2)/4\tau} y \frac{\partial f_{\widehat{\sigma}}}{\partial y}(\overline{u} - z_1, \overline{v} - z_2, 0) dz_1 dz_2 \\ &+ \lim_{\widehat{\sigma}^2 \to \sigma^2} \sigma^2 \frac{f_{\widehat{\sigma}} - f_\sigma}{\widehat{\sigma}^2 - \sigma^2}(x, \sigma^2, T - \Delta t). \end{aligned} \tag{3.3}$$

Using the fact that $\frac{\partial f_{\widehat{\sigma}}}{\partial y} = 0$ at $t = T$ for all x and y, the first term in the right hand side equals zero and we can also easily show that the last term of the above equation is less than $C(\Delta t)^{\frac{3}{2}}$ for some constant C and small Δt. Therefore, choosing $\sigma = \sqrt{y}$, we have that

$$|y\frac{\partial f}{\partial y}(x, y, T - \Delta t)| \leq C(\Delta t)^{\frac{3}{2}} \tag{3.4}$$

for all $x, y \in \mathbb{R}_+$.

For the interval $[T - k\Delta t, T - (k - 1)\Delta t]$, we can complete the proof by the same argument except that $y\frac{\partial f_{\widehat{\sigma}}}{\partial y}$ which is in the first integral of (3.3) is not zero, but bounded by $C(\Delta t)^{\frac{3}{2}}$ due to the previous step at time $t = T - (k - 1)\Delta t$. ∎

Now, we present the mixed second derivative estimate which is needed in the last section.

Lemma 3.3. *Suppose that $f(x, y, t)$ is the solution of the Iterative Problem (1.5). Then, at each $t = T - k\Delta t$ ($k = 1, \cdots, n$), $|xy\frac{\partial^2 f}{\partial x \partial y}(x, y, t)| \leq C(\Delta t)^{\frac{1}{2}}$ for some constant C independent of x, y and Δt.*

Proof. Note that since $x\frac{\partial}{\partial x}(x^2\frac{\partial^2}{\partial x^2}) = x^2\frac{\partial^2}{\partial x^2}(x\frac{\partial}{\partial x})$, the operator $x\frac{\partial}{\partial x}$ commutes with the operator L_σ. Thus, $x\frac{\partial f_\sigma}{\partial x}$ satisfies the equation

$$L_\sigma\left(x\frac{\partial f_\sigma}{\partial x}\right) = 0.$$

We claim that $yx^3\frac{\partial^3 f_\sigma}{\partial x^3}$ is uniformly bounded for $t = T - k\Delta t$ $(k = 1, \cdots, n)$ by a constant independent of x, y and Δt. This can be obtained from the fact that $yx^3\frac{\partial^3 \tilde{f}}{\partial x^3}$ is bounded for all x, y and t as in Lemma 2.1. Thus the argument in Lemma 3.2 leading to (3.4) can be used to complete the proof. ∎

4. Discrete Time Replicating Strategy

In this section, we consider the hedging problem, which is one of the most important issues in trading derivative securities. In the ideal Black-Scholes model, the delta hedging strategy replicates a call option perfectly by continuous trading. However, in practice, it is impossible to trade continuously; one can only devise a discrete time hedging strategy. In this case, the best one can hope for is to have the hedging error reduce to reasonably small.

Another advantage of using $f(x, y, t)$ instead of $\tilde{f}(x, y, t)$ is that the standard delta hedging at a discrete time interval as in Leland [7] is naturally possible. The main result in this section is Theorem 4.1 which says that the hedging error using $f(x, y, t)$ decreases to zero as the hedging interval Δt goes to zero. In what follows, by abuse of language, we call $f(x, y, t)$ which is the solution of the Iterative Problem (1.5) the price of a call option when the stock price is x and the variance is y at time t.

Over the kth time interval $[T - k\Delta t, T - (k-1)\Delta t]$, an asset price process S and its variance V satisfy the following discrete equation

$$\begin{aligned} \frac{\Delta S}{S} &= \phi \Delta t + \sigma_k w \sqrt{\Delta t} \\ \frac{\Delta V}{V} &= \mu \Delta t + \xi z \sqrt{\Delta t} \end{aligned} \tag{4.1}$$

where w, z are normally distributed random variables with mean zero and variance one, and σ_k denotes the value of $\sqrt{V}$ at time $t = T - k\Delta t$. If the interval $[T - k\Delta t, T - (k-1)\Delta t]$ and the time $T - k\Delta t$ are understood in the context, we drop the subscript k from σ_k. Consider a fixed portfolio P consisting of N shares of stock and B dollars of the risk free security over the interval Δt. We revise the portfolio at the beginning of each interval. The length of the interval Δt would be the hedging interval. We define a replicating strategy as the delta hedging given by

$$\begin{aligned} N &= \frac{\partial f}{\partial x} \\ B &= f - x\frac{\partial f}{\partial x}. \end{aligned}$$

In the Black-Scholes world, this delta hedging makes a riskless position. But for the discrete time hedging in the stochastic volatility model, we cannot eliminate all the risk. Therefore, the hedging strategy always generates errors and we need to measure the difference ΔH between the changes in value of the replicating portfolio and of the call option over the period $[T-k\Delta t, T-(k-1)\Delta t]$. Here, we compute the hedging error ΔH inside the small time interval of length Δt where Δt is the revision period. Then, we consider the total sum of the expected hedging errors and the variances of the hedging error over $[0,T]$. Finally, we prove that the total hedging error converges to zero almost surely using the derivative estimates which are proved in Section 3 and the law of large numbers.

Theorem 4.1. *If we follow this replicating strategy* $N=\frac{\partial f}{\partial x},\ B=f-x\frac{\partial f}{\partial x}$ *where* f *is the solution of the Iterative Problem* (1.5), *then we can reduce the hedging error to zero almost surely by letting the hedging interval tend to zero.*

Proof. Over the kth interval $[T-k\Delta t, T-(k-1)\Delta t]$, the return of the replicating portfolio P will be

$$\begin{aligned}\Delta P &= N\Delta x + Br\Delta t + O(\Delta t^2)\\ &= \frac{\partial f}{\partial x}\Delta x + (f-\frac{\partial f}{\partial x}x)r\Delta t + O(\Delta t^2).\end{aligned}$$

Here the term $O(\Delta t^2)$ comes from the continuous compounding of interest. Note that the expression $O(\Delta t)$ hereafter has the following meaning; the random variable X is said to be $O(\Delta t)$ if $\limsup_{\Delta t\to 0}\frac{|E(X)|}{\Delta t}$ is bounded by some constant C independent of Δt. On the other hand,

$$\begin{aligned}\Delta f =& f(x+\Delta x, y+\Delta y, t+\Delta t) - f(x,y,t)\\ =&\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(\Delta y)^2\\ &+\frac{\partial^2 f}{\partial x\partial y}(\Delta x)(\Delta y) + O(\Delta t^{3/2})\end{aligned}$$

by Taylor's theorem. From (4.1), $\Delta x = x(\phi\Delta t + \sigma_k w\sqrt{\Delta t})$ and $\Delta y = y(\mu\Delta t + \xi z\sqrt{\Delta t})$. Thus the x-derivative of f in the above is accompanied with the multiplication by x, similarly for y-derivatives, and higher mixed order derivatives are also multiplied with suitable powers of x or y. Then, considering the hedging error ΔH over the same interval, we get

$$\begin{aligned}\Delta H =& \Delta P - \Delta f\\ =&(f - x\frac{\partial f}{\partial x})r\Delta t - \frac{\partial f}{\partial y}\Delta y - \frac{\partial f}{\partial t}\Delta t - \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 - \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(\Delta y)^2\\ &-\frac{\partial^2 f}{\partial x\partial y}(\Delta x)(\Delta y) + O\left(\Delta t^{3/2}\right).\end{aligned}$$

Since f satisfies

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 f}{\partial x^2} + \frac{1}{2}\xi^2 y^2\frac{\partial^2 f}{\partial y^2} - rf + rx\frac{\partial f}{\partial x} + \mu y\frac{\partial f}{\partial y} = 0$$

where σ is the value of $\sqrt{y}$ at time $t = T - k\Delta t$, substituting for $(f - x\frac{\partial f}{\partial x})r\Delta t$ gives

$$\begin{aligned}\Delta H =& \mu y\frac{\partial f}{\partial y}\Delta t + \frac{\partial f}{\partial t}\Delta t + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 f}{\partial x^2}\Delta t + \frac{1}{2}\xi^2 y^2\frac{\partial^2 f}{\partial y^2}\Delta t - \frac{\partial f}{\partial y}\Delta y - \frac{\partial f}{\partial t}\Delta t\\ & - \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 - \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(\Delta y)^2 - \frac{\partial^2 f}{\partial x\partial y}(\Delta x)(\Delta y) + O\left(\Delta t^{3/2}\right)\\ =& -\xi y\frac{\partial f}{\partial y}z\sqrt{\Delta t} + \frac{1}{2}x^2\frac{\partial^2 f}{\partial x^2}(\sigma^2\Delta t - \sigma^2 w^2\Delta t) + \frac{1}{2}y^2\frac{\partial^2 f}{\partial y^2}(\xi^2\Delta t - \xi^2 z^2\Delta t)\\ & - xy\frac{\partial^2 f}{\partial x\partial y}\sigma\xi wz\Delta t + O\left(\Delta t^{3/2}\right). \end{aligned} \tag{4.2}$$

At this moment, we need to mention some subtle difference on $\frac{\partial f}{\partial y}$ between in (4.2) and in Theorem 3.2. Since the portfolio is rebalanced at the beginning of the interval $[T - k\Delta t, T - (k-1)\Delta t]$ according to the values of x and y at that time, the derivatives for f with respect to y should be considered as the ones for f_σ where σ is the fixed value of $\sqrt{y}$ at time $t = T - k\Delta t$. But the fact $y\frac{\partial f}{\partial y}(x, y, T-(k-1)\Delta t)$ is $C(\Delta t)^{\frac{1}{2}}$ obtained in Theorem 3.2 simply implies that $y\frac{\partial f_\sigma}{\partial y}(x, y, T - k\Delta t)$ is $C(\Delta t)^{\frac{1}{2}}$. Taking expectation, we have

$$E(\Delta H) = O\left(\Delta t^{3/2}\right).$$

Therefore, we can conclude the expected total hedging error when the time to maturity is T

$$E\left(\sum_{k}^{T/\Delta t}\Delta H_k\right) = O(\Delta t^{1/2})$$

which tends to zero as $\Delta t \to 0$. Since $y\frac{\partial f}{\partial y}$ is $C(\Delta t)^{\frac{1}{2}}$ as mentioned above, we obtain

$$E(\Delta H^2) = O(\Delta t^2).$$

To apply Theorem 4.2 [2, p.243], set $X_k = \frac{\Delta H_k}{\Delta t}$, $n = \frac{T}{\Delta t}$ and $b_k = k$. Then, ignoring the term $O\left(\Delta t^{3/2}\right)$, $E(X_k|\mathcal{F}_{(k-1)\Delta t}) = 0$ and $EX_k^2 < C$ for some constant C. Since

$$\sum_{k=1}^{\infty}\frac{EX_k^2}{b_k^2} < C\sum_{k=1}^{\infty}\frac{1}{k^2} < \infty,$$

it follows that

$$\frac{\sum_{k=1}^n X_k}{b_n} = \frac{1}{n}\sum_{k=1}^{n}\frac{\Delta H_k}{\Delta t} = \frac{1}{T}\sum_{k=1}^{n}\Delta H_k \longrightarrow 0 \quad \text{a.s.}$$

which means the total hedging error over the period $[0, T]$ will almost surely be zero as Δt goes to zero. ∎

Theorem 4.2 (The Law of Large Numbers). *Let $X_1, X_2, \cdots$ be a sequence of random variables such that $X_k \in \mathcal{F}_k$ and $E(X_k|\mathcal{F}_{k-1}) = 0$ for all k, where $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ is an increasing sequence of σ-algebras of events. If $b_1 < b_2 < \cdots \to \infty$ and $\sum_{k=1}^{\infty} \frac{EX_k^2}{b_k^2} < \infty$, then*

$$\frac{X_1 + \cdots + X_n}{b_n} \xrightarrow{\text{a.s.}} 0.$$

5. Replicating Strategy under Transaction Costs

In this section, we include transaction costs in the model. If transaction costs are taken into consideration in the stochastic volatility model, it is clear that the contingent claim cannot be replicated perfectly. Therefore, the intrinsic hedging error would occur for any trading strategy, and the valuation and hedging problem becomes more complicated.

We propose a replicating strategy which is similar in Section 4 by just modifying the variance at each revision time, and show that the hedging error tends to zero almost surely as the hedging interval goes to zero. Therefore, the hedging method using our new iterative PDE family also solves the valuation and hedging problem in the stochastic volatility model with transaction costs.

Let ν represent the rate of transaction costs and define

$${\sigma_*}^2 = \sigma^2 \left[1 + \frac{\sqrt{\frac{2}{\pi}}\nu}{\sigma\sqrt{\Delta t}} \right].$$

Let f_* be the solution of the Iterative Problem (1.5) with σ replaced by σ_*. To clarify the argument in this section, we need to write here the definition of f_* as follows.

$$< \textbf{Iterative Problem} > \tag{5.1}$$

Let $f(x, y, T)$ be the boundary condition, or equivalently, the contingent claim. Define $f_*(x, y, T - k\Delta t)$ for $(x, y) \in \mathbb{R}_+^2$ and $k = 1, 2, \cdots, n$, recursively as follows:

Step 1 (Initial Step) For each fixed $y > 0$, let $\sigma = \sqrt{y}$. Define $f_{\sigma_*} = (x, y, t)$ to be the solution of the initial value problem

$$\begin{cases} L_{\sigma_*} u = \frac{\partial u}{\partial t} + \frac{1}{2}\sigma_*^2 x^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\xi^2 y^2 \frac{\partial^2 u}{\partial y^2} - ru + rx\frac{\partial u}{\partial x} + \mu y \frac{\partial u}{\partial y} = 0 \\ u(x, y, T) = f(x, y, T) \end{cases}$$

for $(x, y) \in \mathbb{R}_+^2$ and $t \in [T - \Delta t, T]$. Then set

$$f_*(x, y, T - \Delta t) = f_{\sigma_*}(x, y, T - \Delta t).$$

Step 2 (Recursive Step) Suppose $f_*(x,y,T-(k-1)\Delta t)$ is already found. Then for each fixed $y>0$, let $\sigma=\sqrt{y}$. Define $f_{\sigma_*}(x,y,t)$ to be the solution of the problem

$$\begin{cases} L_{\sigma_*}u = \frac{\partial u}{\partial t} + \frac{1}{2}\sigma_*^2 x^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\xi^2 y^2 \frac{\partial^2 u}{\partial y^2} - ru + rx\frac{\partial u}{\partial x} + \mu y \frac{\partial u}{\partial y} = 0 \\ u(x,y,T-(k-1)\Delta t) = f_*(x,y,T-(k-1)\Delta t) \end{cases}$$

for $(x,y)\in\mathbb{R}_+^2$ and $t\in[T-k\Delta t, T-(k-1)\Delta t]$. Then set

$$f_*(x,y,T-k\Delta t) = f_{\sigma_*}(x,y,T-k\Delta t).$$

Let us first estimate the derivatives for f_* by the slight modification of the arguments in Section 3. Here, we simply skip the same computation procedure as in Section 3 and give a little description in the proof. Recall the meaning of $O(\Delta t)$ given in the middle of the proof of Theorem 4.1.

Lemma 5.1. *Suppose that $f_*(x,y,t)$ is the solution of the Iterative Problem (5.1). Then $x^2\frac{\partial^2 f_*}{\partial x^2}$ is $O(\Delta t^{\frac{1}{2}})$ for $t=T-k\Delta t$ $(k=1,2,\cdots,n)$.*

Proof. At each $t=T-k\Delta t$, fix $\sigma=\sqrt{y}$ and put ${\sigma_*}^2=\sigma^2\left[1+\frac{\sqrt{\frac{2}{\pi}}\nu}{\sigma\sqrt{\Delta t}}\right]$. Then, $L_{\sigma_*}f_{\sigma_*}=0$. Applying the operator ${\sigma_*}^2x^2\frac{\partial^2}{\partial x^2}$ to the both sides of this equation, we get

$$L_{\sigma_*}\left({\sigma_*}^2 x^2 \frac{\partial^2 f_{\sigma_*}}{\partial x^2}\right) = 0.$$

Thus, we can derive using the maximum principle that ${\sigma_*}^2x^2\frac{\partial^2 f_{\sigma_*}}{\partial x^2}$ is $O(1)$ for $t=T-k\Delta t$ $(k=1,2,\cdots,n)$ from the boundary condition as in Lemma 3.1. Hence, ${\sigma_*}^2$ is $O(\Delta t^{-\frac{1}{2}})$ implies that $x^2\frac{\partial^2 f_*}{\partial x^2}$ is $O(\Delta t^{\frac{1}{2}})$. ∎

Lemma 5.2. *If $f_*(x,y,t)$ is the solution of the Iterative Problem (5.1), then $y\frac{\partial f_*}{\partial y}$ and $xy\frac{\partial^2 f_*}{\partial x \partial y}$ are $O(\Delta t^{\frac{1}{2}})$.*

Proof. Let us introduce y', a new variable corresponding to ${\sigma_*}^2$ when $y=\sigma^2$, defined by

$$y' = y\left[1+\frac{\sqrt{\frac{2}{\pi}}\nu}{\sqrt{y\Delta t}}\right].$$

Using the fact that ${\sigma_*}^2x^2\frac{\partial^2}{\partial x^2}f_{\sigma_*}$ is $O(1)$ which is obtained in Lemma 5.1, we can estimate

$${\sigma_*}^2\frac{f_{\widehat{\sigma}_*}-f_{\sigma_*}}{\widehat{\sigma}_*^2-\sigma_*^2}(x,\sigma^2,T-k\Delta t)$$

in the same way as in Lemma 3.2 to derive $y'\frac{\partial f_*}{\partial y'}$ is $O(\Delta t^{\frac{1}{2}})$. However, it is easily checked that

$$y\frac{\partial f_*}{\partial y} = \left(y\frac{dy'}{dy}\right)\frac{\partial f_*}{\partial y'} \leq y'\frac{\partial f_*}{\partial y'}$$

due to $y\frac{dy'}{dy} = y\left[1 + \frac{\sqrt{\frac{2}{\pi}}\nu}{\sqrt{y\Delta t}} - \frac{\sqrt{\frac{2}{\pi}}\nu}{2\sqrt{y\Delta t}}\right] \leq y'$. Therefore, we can conclude $y\frac{\partial f_*}{\partial y}$ is $O(\Delta t^{\frac{1}{2}})$. ∎

Now, we show that the replicating strategy $\{N = \frac{\partial f_*}{\partial x}, B = f_* - x\frac{\partial f_*}{\partial x}\}$ will yield payoff of a call option including transaction costs almost surely as $\Delta t \to 0$. Over the interval $[(k-1)\Delta t, k\Delta t]$, the return, ΔP, of this portfolio consisting of N shares of stocks and B dollars of the risk free security and the change in value, Δf_*, of a call option are computed as in the previous section. Here, note that $x\frac{\partial^2 f_*}{\partial x \partial t}$ is $O(\Delta t^{\frac{1}{2}})$. In fact, since $x\frac{\partial^2 f_*}{\partial x \partial t}$ includes no y-derivative, this estimate can be inherited from the fact $x\frac{\partial^2 C}{\partial x \partial t}$ is $O(\Delta t^{\frac{1}{2}})$ where C is the Black-Scholes price with the volatility σ_*. Also, the number of assets bought or sold is

$$\begin{aligned}\Delta N &= \frac{\partial f_*}{\partial x}(x+\Delta x, y+\Delta y, t+\Delta t) - \frac{\partial f_*}{\partial x}(x,y,t) \\ &= \frac{\partial^2 f_*}{\partial x^2}(x,y,t)\Delta x + \frac{\partial^2 f_*}{\partial x \partial y}(x,y,t)\Delta y + \frac{\partial^2 f_*}{\partial x \partial t}\Delta t + O\left(\Delta t^{3/2}\right)\end{aligned}$$

and the transaction costs in this interval would be

$$\begin{aligned}\nu(x+\Delta x)|\Delta N| &= \nu(x+\Delta x)|\frac{\partial^2 f_*}{\partial x^2}\Delta x| + O\left(\Delta t^{3/2}\right) \\ &= \nu x^2\frac{\partial^2 f_*}{\partial x^2}\left|\frac{\Delta x}{x}\right| + O\left(\Delta t^{3/2}\right).\end{aligned}$$

Then, by easy calculation the hedging error inclusive of transaction costs is

$$\begin{aligned}\Delta H =& \Delta P - \Delta f_* - \text{ transaction costs} \\ =& -\xi y\frac{\partial f_*}{\partial y}z\sqrt{\Delta t} + \frac{1}{2}x^2\frac{\partial^2 f_*}{\partial x^2}\left({\sigma_*}^2\Delta t - \sigma^2 w^2\Delta t - \nu\left|\frac{\Delta x}{x}\right|\right) \\ &+ \frac{1}{2}y^2\frac{\partial^2 f_*}{\partial y^2}(\xi^2\Delta t - \xi^2 z^2\Delta t) - \frac{\partial^2 f_*}{\partial x \partial y}(\Delta x)(\Delta y) + O\left(\Delta t^{3/2}\right) \\ =& -\xi y\frac{\partial f_*}{\partial y}z\sqrt{\Delta t} + \frac{1}{2}x^2\frac{\partial^2 f_*}{\partial x^2}\left(\sigma^2\Delta t + \sqrt{\frac{2}{\pi}}\nu\sigma\sqrt{\Delta t} - \sigma^2 w^2\Delta t - \nu\sigma|w|\sqrt{\Delta t}\right) \\ &+ \frac{1}{2}y^2\frac{\partial^2 f_*}{\partial y^2}(\xi^2\Delta t - \xi^2 z^2\Delta t) - xy\frac{\partial^2 f_*}{\partial x \partial y}\sigma\xi w z\Delta t + O\left(\Delta t^{3/2}\right).\end{aligned}$$

Taking expectation, we have $E(\Delta H) = O\left(\Delta t^{3/2}\right)$ and the expected total hedging error $E\left(\sum_k^{T/\Delta t}\Delta H_k\right) = O\left(\Delta t^{1/2}\right)$ which tends to zero as $\Delta t \to 0$. Note that $x^2\frac{\partial^2 f_*}{\partial x^2}$ is $O\left(\Delta t^{\frac{1}{2}}\right)$, which implies $E(\Delta H^2) = O\left(\Delta t^2\right)$. Therefore, Theorem 4.2 implies the total hedging error over $[0,T]$ tends to zero almost surely. Hence, we complete the proof of the following Theorem.

Theorem 5.3. *If we follow the replicating strategy* $\{\frac{\partial f_*}{\partial x},\ f_* - x\frac{\partial f_*}{\partial x}\}$ *where* f_* *is the solution of the Iterative Problem* (5.1)*, then we can reduce the hedging error including transaction costs to zero almost surely as* $\Delta t \to 0$.

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