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ARTICLE

Option valuation with liquidity risk and jumps

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ABSTRACT
This article provides a simple model for pricing and hedging options in the presence of jumps and liquidity costs. In the article, liquidity risk is modelled via a stochastic supply curve function and a jump-diffusion process is approximated by a Markov chain. Local risk minimization incorporating liquidity risk is proposed to price and hedge European options in this discrete-time model. Moreover, an example is provided to implement the modified risk minimization method and to demonstrate the performance of hedging strategies.

KEYWORDS
Option pricing; jump-diffusion model; liquidity risk; local risk minimization

JEL CLASSIFICATION
G13; G19; C65

I. Introduction

Liquidity risk is the risk from the timing and size of a trade, that is, extra cost due to the absence of a counter party. A given security or asset cannot be traded quickly enough to meet the short-term financial demands of the holder under liquidity risk which is considered as a most important risk these days in addition to market risk and credit risk, especially since the financial crisis in 2008. In a market with liquidity risk, investors cannot buy or sell large quantities of security at a given market price, and there must be extra cost associated with buying or selling a given security. The extra cost is regarded as liquidity cost and it typically depends on both the securities market price and trading volume (or trading speed). The pricing and hedging problem of derivatives under liquidity risk has become an important and difficult question in recent years.

Cetin, Jarrow, and Protter (2004) proposed a rigorous model incorporating liquidity risk into the arbitrage pricing theory. Based on this model, Cetin, Soner, and Touzi (2010) used strategies with minimal super-replication cost inclusive of liquidity premium to price contingent claims in continuous time setting. Ku, Lee, and Zhu (2012) derived a partial differential equation which provides discrete-time delta hedging strategies whose expected hedging errors approach zero almost surely as the length of the revision interval goes to zero (see also Sorokin and Ku, 2016). In these papers, the stock price is assumed to follow a geometric Brownian motion. However, evidence of jumps in the stock price has been provided by empirical studies of stock return (see, for example, Jorion 1988; Bates 2000). When there are jumps in the underlying asset, liquidity risk becomes a critical problem. Recently, Lehman Brother's collapse gave us a concrete example of the dramatic consequence of combining jump risk and liquidity risk.

In this article, we investigate option valuation with liquidity risk in a jump-diffusion model. Jumps in stock price bring jump risk, and it is known that liquidity risk and jump risk are not independent, but are correlated. Jump risk has been an important topic in the pricing and hedging of contingent claims since Merton (1976). In a financial crisis, it is common that an underlying asset price exhibits jumps, leading investors in the market to change their positions quickly on the underlying asset to hedge derivatives, which causes a significant liquidity problem. The severity of combining jumps and liquidity risk occurs in these situations. Therefore, the pricing and hedging problem in a jump-diffusion model under liquidity costs is an important practical question.

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When the underlying asset follows a jump-diffusion process, a market is incomplete and a contingent claim cannot be replicated with the underlying. There are some approaches to price derivatives in an incomplete market, for instance, super-hedging, mean-variance hedging and local risk minimization approach (see, for example, Lim 2005; Follmer and Schweizer 1991; Coleman, Li, and Patron 2007). Local risk minimization is an easily applicable method to price options in incomplete markets. One can price options by the local risk minimization method for a jump-diffusion model without liquidity risk in the continuous time setting, which gives us a partial differential equation to characterize the initial hedging cost. It is natural to ask whether one can derive a modified partial differential equation to describe the local risk minimization hedging cost of options in a market with liquidity risk. It does not seem possible to derive such a partial differential equation due to the complexity introduced by liquidity risk. Thus, we address and investigate this issue in discrete time.

A jump-diffusion model has been approximated by a discrete-time process in the literature (see, for instance, Amin 1993). In this article, we apply local risk minimization to price options with liquidity risk for a Markov chain converging in distribution to a continuous jump-diffusion process. Therefore, the option price obtained from the discrete-time model approaches the option price in the jump-diffusion model as the time step goes to zero. Hence, the method proposed in this article provides a valuation and hedging model for options in the presence of jumps and liquidity costs.

The article is organized as follows. Section II is devoted to introduce a discrete-time Markov process which approximates the jump diffusion process, and shows the proof of the convergence. Section III discusses the local risk minimization method including liquidity risk in our discrete-time model. Section IV presents some numerical results and Section V concludes the article.

II. Markov chain approximation of a jump-diffusion model

In this section, we present the local risk minimization method for a jump-diffusion process without liquidity risk. We consider a financial market which consists of a risk-free asset and a risky asset. The money market account B_t with the risk-free rate r is given by

$$dB_t = rB_t\,dt, t \in [0, T]$$

Without loss of generality, it is assumed that $r = 0$. The asset price is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $\{\mathrm{F}_t : t \geq 0\}$ generated by a one-dimensional Brownian motion W_t and a Poisson process N_t with intensity λ. The stock price S_t is modelled by a jump-diffusion process that follows the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t + (V_i - 1)S_t dN_t, \quad t \in [0, T] \tag{1}$$

where σ is the volatility, μ is the drift term of the stock and V_i is the jump size where

$$\mathbb{P}\{V_i = e^{q_j}\} = p_j, 1 \leq j \leq m$$

and

$$p_1 + p_1 + \ldots + p_m = 1$$

The solution for the stochastic differential Equation 1 is written as

$$S_t = S_0 \exp\left\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right\} \prod_{i=1}^{N(t)} V_i$$

It is well known that a geometric Brownian motion is approximated by a binomial model. If the jump size takes finitely many possible values, a jump-diffusion process can be approximated by a discrete-time process. In the following, we present a Markov chain approximation of the jump-diffusion model.

We approximate the jump-diffusion process (Equation 1) in the following way. Let $N(t)$ be a Poisson process with intensity λ. For any $t \in [0, T]$, we have N stages over time horizon $[0, t]$, denoted by $0 = t_0 < t_1 < \cdots < t_N = t$ with $\Delta t = \frac{t}{N}$. Given S_k, the stock price at time t_k, and time step Δt, there are $m + 2$ possible values for S_{k+1} at time $k + 1$:

$$S_{k+1} = \begin{cases} S_k e^{(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}}, & \text{if } \mathrm{S_k} \text{ goes up} \\ S_k e^{(\mu - \frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}}, & \text{if } \mathrm{S_k} \text{ goes down} \\ e^{q_1} S_k, & \text{if } \mathrm{S_k} \text{ jumps to } e^{q_1} S_k \\ \cdots & \\ e^{q_m} S_k, & \text{if } \mathrm{S_k} \text{ jumps to } e^{q_m} S_k \end{cases}$$

The relationship between S_{k+1} and S_k is $S_{k+1} = S_k\xi_{k+1}$, where

$$\xi_{k+1} = \begin{cases} e^{(\mu-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}}, & \text{with probability } \frac{1-\lambda\Delta t}{2} \\ e^{(\mu-\frac{1}{2}\sigma^2)\Delta t-\sigma\sqrt{\Delta t}}, & \text{with probability } \frac{1-\lambda\Delta t}{2} \\ e^{q_1}, & \text{with probability } p_1\lambda\Delta t \\ \cdots & \\ e^{q_m}, & \text{with probability } p_m\lambda\Delta t \end{cases}$$

Theorem 2.1. *As $N \to \infty$, the distribution of $(S_k)_{k=0,1,\ldots,N}$ converges to the distribution of*

$$S_t = S_0 \exp\left\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\right\}\prod_{i=1}^{N(t)} V_i$$

where $\mathbb{P}\{V_i = e^{q_j}\} = p_j$ for $1 \le i \le N(t), 1 \le j \le m$ and $p_1 + p_2 + \ldots + p_m = 1$.

Proof. Notice that $S_{k+1} = S_k\xi_{k+1}$, then S_N can be written as

$$S_N = S_0\xi_1\xi_2\ldots\xi_N$$

Denoting $\eta_k = \ln(\xi_k)$ and $X_N = \ln\left(\frac{S_N}{S_0}\right)$, we shall have

$$S_N = S_0 e^{\eta_1+\eta_2+\ldots+\eta_N}$$

For this discrete-time model, the log return X_N has the form of

$$X_N = \eta_1 + \eta_2 + \ldots + \eta_N$$

where $\eta_1, \eta_2, \ldots\eta_N$ are independent and identically distributed.

For the continuous time jump-diffusion model, the log return $X_t = \ln\left(\frac{S_t}{S_0}\right)$ is expressed as

$$X_t = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t + \sum_{i=1}^{N(t)} U_i$$

where $U_i = \ln(V_i)$. The moment-generating function of X_t is

$$\begin{aligned} G_{X_t}(\theta) &= E\left[e^{\theta X_t}\right] \\ &= E\left[e^{\theta[(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t+\sum_{i=1}^{N(t)} U_i]}\right] \\ &= e^{\theta(\mu-\frac{1}{2}\sigma^2)t}E\left[e^{\theta\sigma W_t}\right]E\left[e^{\theta\sum_{i=1}^{N(t)} U_i}\right] \end{aligned}$$

By the iterated conditional expectation, we have

$$\begin{aligned} E\left[e^{\theta\sum_{i=1}^{N(t)} U_i}\right] &= E\left\{E\left[e^{\theta\sum_{i=1}^{N(t)} U_i}|N(t)\right]\right\} \\ &= E\left[\left(p_1e^{q_1\theta} + p_2e^{q_2\theta} + \ldots + p_me^{q_m\theta}\right)^{N(t)}\right] \\ &= \sum_{k=0}^{\infty}\left(p_1e^{q_1\theta} + p_2e^{q_2\theta} + \ldots + p_me^{q_m\theta}\right)^k\frac{(\lambda t)^k e^{-\lambda t}}{k!} \\ &= \exp\left\{\lambda\left(p_1e^{q_1\theta} + p_2e^{q_2\theta} + \ldots + p_me^{q_m\theta} - 1\right)t\right\} \end{aligned} \tag{2}$$

Also, we know

$$E\left[e^{\theta\sigma W_t}\right] = \exp\left(\frac{1}{2}\sigma^2\theta^2 t\right) \tag{3}$$

Together with Equations 2 and 3, the moment-generating function $G_{X_t}(\theta)$ is expressed as

$$\begin{aligned} G_{X_t}(\theta) = \exp\Big\{&\theta\left(\mu - \frac{1}{2}\sigma^2\right)t + \frac{1}{2}\sigma^2\theta^2 t \\ &+ \lambda\left(p_1e^{q_1\theta} + p_2e^{q_2\theta} + \ldots + p_me^{q_m\theta} - 1\right)t\Big\} \end{aligned}$$

Then, $G_{X_N}(\theta) = E[e^{\theta X_N}]$ is written as

$$\begin{aligned} &G_{X_N}(\theta) \\ &= \left[G_{\eta_1}(\theta)\right]^N \\ &= \left\{\frac{1-\lambda\Delta t}{2}e^{\theta[(\mu-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}]} + \frac{1-\lambda\Delta t}{2}e^{\theta[(\mu-\frac{1}{2}\sigma^2)\Delta t-\sigma\sqrt{\Delta t}]}\right. \\ &\qquad \left. + p_1\lambda\Delta t e^{\theta q_1} + \ldots + \mathrm{p_m}\lambda\Delta t e^{\theta \mathrm{q_m}}\right\}^N \\ &= \left\{\frac{1-\lambda\Delta t}{2}\left[1 + \theta(\mu - \frac{1}{2}\sigma^2)\Delta t\right.\right. \\ &\qquad \left. + \sigma\sqrt{\Delta t}\theta + \frac{1}{2}\sigma^2\Delta t\theta^2 + \mathrm{O}(\Delta t)^{3/2}\right] \\ &\qquad + \frac{1-\lambda\Delta t}{2}\left[1 + \theta(\mu - \frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}\theta + \frac{1}{2}\sigma^2\Delta t\theta^2\right. \\ &\qquad \left.\left. + \mathrm{O}(\Delta t)^{3/2}\right] + p_1\lambda\Delta t e^{\theta q_1} + \ldots + \mathrm{p}_m\lambda\Delta t e^{\theta q_m}\right\}^N \\ &= \left\{1 + \theta(\mu - \frac{1}{2}\sigma^2)\Delta t + \frac{1}{2}\sigma^2\theta^2\Delta t + \lambda(p_1e^{\theta q_1}\right. \\ &\qquad \left. + p_2e^{\theta q_2} + \ldots p_me^{\theta q_m} - 1)\Delta t + \mathrm{O}(\Delta t)^{3/2}\right\}^N \end{aligned}$$

$$= \left\{1 + \left[\theta(\mu - \frac{1}{2}\sigma^2) + \frac{1}{2}\sigma^2\theta^2 + \lambda(p_1 e^{q_1\theta} + p_2 e^{q_2\theta} + \ldots + p_m e^{q_m\theta} - 1)\right]\Delta t + O(\Delta t)^{3/2}\right\}^N$$

Since $N = \frac{t}{\Delta t}$, as $N \to \infty$ we have

$$\lim_{N\to\infty}\left\{1 + \left[\theta(\mu - \frac{1}{2}\sigma^2) + \frac{1}{2}\sigma^2\theta^2 + \lambda(p_1 e^{q_1\theta} + p_2 e^{q_2\theta} + \ldots + p_m e^{q_m\theta} - 1)\right]\Delta t + O(\Delta t)^{3/2}\right\}^N$$

$$= \lim_{\Delta t\to 0}\left\{1 + \left[\theta(\mu - \frac{1}{2}\sigma^2) + \frac{1}{2}\sigma^2\theta^2 + \lambda(p_1 e^{q_1\theta} + p_2 e^{q_2\theta} + \ldots + p_m e^{q_m\theta} - 1)\right]\Delta t + O(\Delta t)^{3/2}\right\}^{\frac{t}{\Delta t}}$$

$$= \exp\left\{\theta(\mu - \frac{1}{2}\sigma^2)t + \frac{1}{2}\sigma^2\theta^2 t + \lambda(p_1 e^{q_1\theta} + p_2 e^{q_2\theta} + \ldots + p_m e^{q_m\theta} - 1)t\right\}$$

which is exactly the generating function of X_t. We proved the moment-generating function of X_N converges to that of X_t. Therefore, $S_N = S_0 e^{X_N}$ converges to $S_t = S_0 e^{X_t}$ in distribution as desired.

III. Local risk minimization under liquidity risk

In this section, liquidity risk is described by a stochastic supply curve. A supply curve function $S(t, z, \omega)$ represents the stock price per share that the investor pays/receives for an order size of $z \in R$ at time t. A positive z represents a purchase order and a negative z represents a sale order of stock. The supply curve function is determined by the market structure; therefore, a single investor's past actions, wealth and risk attitude have no impact on the supply curve. It is generally believed that the supply curve satisfies the following assumptions:

- $S(t, z, \omega)$ is $\mathcal{F}_t$ measurable and nonnegative.
- $S(t, z, \omega)$ is nondecreasing in z.
- $S(t, z, \omega)$ is continuous in z.

From now on, we write this stochastic supply curve function $S(t, z, \omega)$ as $S_t(z)$ in order to simplify the notation.

Due to the liquidity risk, investors face the fact of selling at a lower price than the market-quoted price and buying at a higher price than the market-quoted price. Therefore, liquidity risk adds extra cost for trading which is regarded as liquidity cost. We assume the supply curve function is in the separable form as in Ku, Lee, and Zhu (2012), which is given by

$$S_t(z) = f(z)S_t$$

where $f(\cdot)$ is a positive, continuous and nondecreasing function with $f(0) = 1$, and S_t is the quoted price (mid-price) at time t.

Based on theoretical developments in Section II, we use a discrete-time approximation for the asset price, and address the pricing and hedging problem for discrete-time process. When the time interval $\Delta t (= \frac{t}{N})$ goes to zero, the option price obtained from the discrete-time model converges to the option price including liquidity costs in the jump-diffusion model.

Assume we are going to hedge a European call option with maturity t_N and pay-off $H_N = (S_N - K)^+$ which is $\mathcal{F}_{t_N}$ measurable. A trading strategy is given by two stochastic processes $(x_k)_{k=0,1,\ldots,N}$ and $(y_k)_{k=0,1,\ldots,N}$, where x_k stands for the number of shares of the asset S_k held and y_k is the amount in the money market account at time t_k. Both x_k and y_k are $\mathcal{F}_{t_k}$ measurable for $0 \le k \le N$. The portfolio is a combination of the stock and money market account for the trading strategy. The value of portfolio (the marked-to market value) at time t_k is given by

$$V_k = x_k S_k + y_k$$

For $k = 1, 2, \ldots, N$, the liquidity cost incurred from t_1 to t_k is defined by

$$L_k = \sum_{i=0}^{k-1}[f(x_{i+1} - x_i) - 1]S_{i+1}(x_{i+1} - x_i)$$

and the accumulated gain G_k up to time t_k is given by

$$G_k = \sum_{i=0}^{k-1} x_i(S_{i+1} - S_i) - \sum_{i=0}^{k-1}[f(x_{i+1} - x_i) - 1]S_{i+1}(x_{i+1} - x_i)$$

and $G_0 = 0$. Indeed, the accumulated gain in the market with liquidity costs equals the accumulated gain from the changes in stock price minus the accumulated liquidity costs. The accumulated cost at time t_k is defined by

$$C_k = V_k - G_k$$

A strategy is said to be self-financing if the accumulated cost process $(C_k)_{k=0,1,\dots,N}$ is constant over time. This implies

$$\begin{aligned} C_{k+1} - C_k &= (V_{k+1} - G_{k+1}) - (V_k - G_k) \\ &= x_{k+1}S_{k+1} + y_{k+1} + [f(x_{i+1} - x_i) - 1] \\ &\quad S_{i+1}(x_{i+1} - x_i) - x_k S_{k+1} - y_k = 0 \end{aligned}$$

Note that the value of a self-financing portfolio at time t_k is $V_k = V_0 + G_k$ for $0 \le k \le N$. If the market is complete and perfect, there exists a self-financing strategy that satisfies $V_N = H_N$. But if the market is incomplete, a contingent claim can be nonattainable and there may be no hedging strategy under which the cost process $(C_k)_{k=1,2,\dots,N}$ is constant. A hedging strategy needs to be chosen based on some optimality criteria.

Now, we apply the local risk minimization hedging method to hedge options in the discrete-time model. First, we let $V_N = H_N$. Local risk minimization requires the cost process $(C_k)_{k=1,2,\dots,N}$ to be a martingale and the variance of incremental cost process $(C_{k+1} - C_k)_{k=0,1,\dots,N-1}$ to be minimal. Therefore, the traditional criterion for local risk minimization is

$$\text{Minimize Var } [(C_{k+1} - C_k|\mathcal{F}_k]$$

$$\text{Subject to } E[C_{k+1} - C_k|\mathcal{F}_k] = 0$$

which is equivalent to minimize

$$E[(C_{k+1} - C_k)^2|\mathcal{F}_k]$$

In our discrete-time model, given the pay-off H_N at maturity of the option, we set $V_N = x_N S_N + y_N = H_N$. By the local risk minimization method, the trading strategy (x^*_{N-1}, y^*_{N-1}) at t_{N-1} is calculated by

$$(x^*_{N-1}, y^*_{N-1}) = \arg \min_{x_{N-1}, y_{N-1}} E[(H_N - x_{N-1}S_N - y_{N-1})^2|\mathcal{F}_{N-1}]$$

For $0 \le k < N-1$, given the values for (x^*_{k+1}, y^*_{k+1}), we need to minimize $E[(C_{k+1} - C_k)^2|\mathcal{F}_k]$ to determine (x^*_k, y^*_k). It can be done by minimizing the following optimization problem:

$$\begin{aligned} (x^*_k, y^*_k) = \arg \min_{x_k, y_k} E[(&x^*_{k+1}S_{k+1} + y^*_{k+1} \\ &+ \left[f(x^*_{k+1} - x_k) - 1\right] S_{k+1}(x^*_{k+1} - x_k) \\ &- x_k S_{k+1} - y_k)^2|\mathcal{F}_k] \end{aligned}$$

By backward induction, we have (x^*_{N-1}, y^*_{N-1}), $(x^*_{N-2}, y^*_{N-2})\dots,(x^*_1, y^*_1)$, and (x^*_0, y^*_0), recursively. Then the initial option price at time t_0 is determined by the value $x^*_0 S_0 + y^*_0$, and also (x^*_{N-1}, y^*_{N-1}), $(x^*_{N-2}, y^*_{N-2})\dots,(x^*_1, y^*_1)$, (x^*_0, y^*_0) provide the local risk minimization hedging strategies. As N goes to infinity, the discrete-time model converges to the jump-diffusion model. The option price and hedging strategy obtained from the discrete-time model give a good approximation to the corresponding price and hedging strategy in the jump-diffusion model.

It is noted that the discrete model presented in Sections II and III can be viewed as a generalization to the classical binomial model. When the liquidity parameter is 0 (the supply curve function is flat everywhere), our approach coincides with the discrete-time model of a jump-diffusion process with local risk minimization hedging. Also, when the jump parameter $\lambda = 0$ (there are no jumps), our model is reduced to the binomial model with liquidity costs, which is a discrete-time version of a continuous perfect replication model. It is obvious that our model reduces to the classical binomial model when both parameters are 0.

IV. Numerical results

In this section, we present an example for the implementation of the model and show a comparison of numerical experiments on three hedging methods: delta hedging, conventional local risk minimization without liquidity risk and modified local risk minimization (including liquidity costs). First we describe a Markov chain that approximates a jump-diffusion process.

The jump-diffusion model we are going to approximate is given by

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t + (V_i - 1)S_t \mathrm{d}N_t, t \in [0, T]$$

and N_t is a Poisson process with intensity $\lambda_1 + \lambda_2$. For simplicity, we assume that V_i can take two possible values such that

$$\mathbb{P}\{V_i = e^{q_1}\} = \frac{\lambda_1}{\lambda_1 + \lambda_2} \text{ and } \mathbb{P}\{V_i = e^{q_2}\} = \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

Table 1. Option prices with different strikes and volatilities.

	Strike								
Volatility	95	96	97	98	99	100	101	102	103
0.10	10.8885	10.3317	9.7946	9.2755	8.7754	8.2944	7.8312	7.3869	6.9605
0.15	12.2087	11.6831	11.1751	10.6824	10.2043	9.7432	9.2979	8.8699	8.4532
0.20	13.7460	13.2437	12.7540	12.2798	11.8187	11.3702	10.9373	10.5153	10.1065
0.25	15.3913	14.9088	14.4325	13.9696	13.5198	13.0828	12.6587	12.2483	11.8482
0.30	17.0910	16.6219	16.1636	15.7159	15.2788	14.8515	14.4344	14.0270	13.6295

Table 2. Option prices with different values of λ_1 and λ_2.

	λ_2				
λ_1	0	0.25	0.50	0.75	1.00
	9.5957	9.6390	9.6990	9.7583	9.8134
0.25	9.9215	10.0109	10.1081	10.1981	10.2795
0.50	10.2177	10.3445	10.4710	10.5861	10.6897
0.75	10.4771	10.6361	10.7890	10.9266	11.0504
0.00	10.7055	10.8932	11.0694	11.2279	11.3702

The discrete-time model used to approximate the jump-diffusion model has N periods with time step $\Delta t = \frac{T}{N}$. Suppose the stock price at period $k(0 \le k \le N-1)$ is S_k in the discrete-time model, then the stock price at period $k+1$ has four scenarios; it goes up, goes down, jumps down or jumps up. The probability distribution for S_{k+1} is written as

$$S_{k+1} = \begin{cases} S_k(1+\mu\Delta t + \sigma\sqrt{\Delta t}), & \text{with probability} \frac{1-\lambda_1\Delta t-\lambda_2\Delta t}{2} \\ S_k(1+\mu\Delta t - \sigma\sqrt{\Delta t}), & \text{with probability} \frac{1-\lambda_1\Delta t-\lambda_2\Delta t}{2} \\ e^{q_1}S_k, & \text{with probability} \lambda_1\Delta t \\ e^{q_2}S_k, & \text{with probability} \lambda_2\Delta t \end{cases}$$

As the time step $\Delta t \to 0$, this discrete-time Markov process converges to the continuous jump-diffusion process. We use the values $q_1 = \ln 0.9$ and $q_2 = \ln 1.12$ in the computation. We also assume that the supply curve function $f(\cdot)$ is linear and has the following form:

$$S_k(z) = (1+\alpha z)S_k$$

and the hedging error is computed by

$$x_N S_N + y_N - H_N$$

Table 1 presents the European call option price with different strikes and volatilities. The parameter values in this computation are $S_0 = 100$, $T = 1$, $\mu = 0.2$, $\alpha = 0.1$, $\lambda_1 = 1$, $\lambda_2 = 1$ and $N = 50$. Table 2 presents European call option prices with varying λ_1 and λ_2 when $\sigma = 0.2$ and $K = 100$.

Table 3 shows the analysis on the hedging error for a European call option with $\sigma = 0.2$, $K = 100$, $T = 1$ and varying λ_1 and λ_2. Here, *Delta* refers to delta hedging, *LRM* refers to conventional local risk minimization and *MLRM* refers to the modified local risk minimization. *Cost* refers to the mean cost needed to make the hedging error have zero expectation, *Std* is the SD of hedging error and *Liq cost* refers to the mean liquidity cost of the strategies. When we compare the hedging cost of the three different hedging methods, the mean hedging cost of our modified hedging strategy is less than those of the delta hedging strategy and the conventional local risk minimization hedging strategy. More importantly, compared with delta hedging and conventional local risk minimization, the hedging strategy under the modified local risk minimization reduces

Table 3. Analysis on the hedging error under different hedging methods.

		Delta			*LRM*			*MLRM*		
λ_1	λ_2	*Cost*	*Std*	*Liq cost*	*Cost*	*Std*	*Liq cost*	*Cost*	*Std*	*Liq cost*
0.0	0.0	9.8149	1.4773	1.9347	9.7508	1.4291	1.8706	9.5957	0	1.5119
0.5	0.5	10.781	2.0050	1.8960	10.7442	1.9669	1.8768	10.4719	1.3408	1.4221
0.5	1.0	10.896	2.0919	1.7737	10.8692	2.0650	1.7289	10.6951	1.4731	1.3358
1.0	0.5	11.486	2.1329	2.0186	11.4834	2.1694	2.0231	11.0667	1.5709	1.5318
1.0	1.0	11.864	2.3914	1.9873	11.7065	2.2253	1.9024	11.3645	1.6480	1.4713

Table 4. Analysis on the hedging error under different hedging methods.

		Delta			*LRM*			*MLRM*		
λ_1	λ_2	*Cost*	*Std*	*Liq cost*	*Cost*	*Std*	*Liq cost*	*Cost*	*Std*	*Liq cost*
0.0	0.0	7.7334	1.5440	2.1810	7.7298	1.5624	2.1774	7.2883	0	1.6230
0.5	0.5	8.2393	1.9760	1.9707	8.2348	2.0001	1.9668	7.7515	1.3948	1.4013
0.5	1.0	8.2838	2.0948	1.8509	8.3077	2.1738	1.8655	7.8874	1.5775	1.3594
1.0	0.5	8.6712	2.1207	1.9683	8.6248	2.1165	1.9584	8.1468	1.6130	1.4275
1.0	1.0	8.7941	2.2673	1.8838	8.7709	2.2467	1.8740	8.3870	1.8004	1.3953

the SD of hedging error significantly. We thus conclude that among the three hedging strategies, our modified local risk minimization method outperforms the other two hedging methods. Table 4 also shows the hedging error analysis when $\sigma = 0.2$ and $T = 0.5$.

V. Conclusion

We used a Markov chain approximation of a jump model and computed the initial cost for an option by minimizing quadratic incremental costs and solving recursively. We applied the local risk minimization method incorporating liquidity risk to price European options in the discrete-time model with the presence of jumps and liquidity costs. Numerical results showed that the proposed hedging strategies reduce the SD of the hedging error as well as the mean hedging cost, which confirmed that our modified local risk minimization method performs better than other existing hedging methods. Management of risks in combining jump risk and liquidity risk is challenging. This article provided a simple and useful model for option valuation in the presence of jumps and liquidity costs.

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