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Portfolio optimization for a large investor under partial information and price impact

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Abstract This paper studies portfolio optimization problems in a market with partial information and price impact. We consider a large investor with an objective of expected utility maximization from terminal wealth. The drift of the underlying price process is modeled as a diffusion affected by a continuous-time Markov chain and the actions of the large investor. Using the stochastic filtering theory, we reduce the optimal control problem under partial information to the one with complete observation. For logarithmic and power utility cases we solve the utility maximization problem explicitly and we obtain optimal investment strategies in the feedback form. We compare the value functions to those for the case without price impact in Bäuerle and Rieder (IEEE Trans Autom Control 49(3):442–447, 2004) and Bäuerle and Rieder (J Appl Prob 362–378, 2005). It turns out that the investor would be better off due to the presence of a price impact both in complete-information and partial-information settings. Moreover, the presence of the price impact results in a shift, which depends on the distance to final time and on the state of the filter, on the optimal control strategy.

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1 Introduction

In the Merton's classical portfolio optimization problem the underlying stock price process is assumed to be independent of the actions of the investor. When the investor is large, such as hedge funds, mutual funds or insurance companies, this may not hold, as the action of the investor may influence the risky price process through different channels. This gives us a motivation to consider stock price dynamics whose drift is affected by the decisions of the large investor. Moreover, during long investment periods drift of the price process may change in accordance with the changing market conditions. This phenomenon can be reflected by a regime-switching drift. Investors may or may not observe the part of the drift which is governed by the current market conditions.

In this paper, we deal with a finite-time portfolio optimization problem for a large investor, where the underlying price process is a diffusion affected by the state of the market and the actions of the investor. The state of the market is represented by a finite-state Markov chain allowing for the drift of the price process to change in accordance with the changing market conditions for long investment periods. We also let the drift change according to the investment decisions of the large investor. In fact, the drift of the price process is taken to be a function of the fraction of the wealth invested in the risky asset by the large investor. Hence, the portfolio decomposition of the large investor might be considered as another factor governing the drift of the price process.

We first assume that the state of the economy is observable *(complete- information case)* by the investor. By allowing for the price impact, we extend the setting given in Bäuerle and Rieder (2004). Under the full information setting we obtain results for general impact and utility function (logarithmic and power) choices. For any sufficiently regular impact function and for the choice of logarithmic utility, the corresponding optimization problem can be solved directly and optimal investment strategies are characterized explicitly. In the case of power utility, we address the problem by using dynamic programming methods. In particular, the case with linear impact function yields an optimal control in the feedback form as well as a probabilistic representation for the corresponding value function. We show that for both logarithmic and power utility preferences and linear impact function choice the resulting value function dominates the value function corresponding to the utility maximization problem in a setting without price impact.

Secondly, we repeat our analysis for the setting where the state of the economy is not directly observable by the investor *(partial-information case)*. Technically, this results in an optimization problem under partial information. To solve such a problem we derive an equivalent control problem under full information via the so-called reduction approach (see, e.g., Fleming and Pardoux 1982). This requires the derivation of the filtering equation for the unobservable state variable. In order to obtain the filtering equations we address innovations approach to non-linear filtering (see, e.g., Elliott

1982). Then, we introduce the filter for the Markov chain as an additional state variable for the optimization problem. This reduces our problem to a control problem with complete information. We treat logarithmic and power utility cases in the same way explained above. In particular, for the case with linear impact function we obtain an optimal control in the feedback form and we provide a probabilistic representation for the corresponding value function both for logarithmic utility and power utility preferences. We obtain the result for the power utility case by employing a power change of variable approach (see, e.g., Zariphopoulou 2001). Also, we derive no-arbitrage conditions on the impact function.

Overall, we find that the investor benefits from the presence of the price impact in the sense that the resulting value functions corresponding to full-information and partial-information settings dominate the ones given in Bäuerle and Rieder (2004) and Bäuerle and Rieder (2005), respectively. This result, of course, would be counter-intuitive for the problem of an optimal order execution, where the investor is allowed to trade only in one direction, e.g. only buy or sell orders. Moreover, our numerical results based on a two-state Markov chain suggest that the presence of the price impact yields a shift on the optimal control strategies. In particular, for the case of partial information the magnitude of the shift depends on the distance to final time and on the state of the filter.

There is an ample amount of literature concerning the portfolio optimization problems with Markov chain modulated price dynamics under complete and partial-information. The case with complete information has been addressed in Bäuerle and Rieder (2004), in which the problem of expected utility maximization from terminal wealth is solved by stochastic control methods for different utility functions. Sass and Haussmann (2004) and Haussmann and Sass (2004) have treated the portfolio optimization problem in a multi-asset setting under partial information and they obtained the optimal portfolio strategy by using the martingale approach. On the other hand, Bäuerle and Rieder (2005) has addressed the portfolio optimization problem with unobservable Markov chain modulated drift process by using a dynamic programming approach. Björk et al. (2010) considers a relatively general setting and it provides explicit representations of the optimal wealth and investment processes for the utility maximization problem under partial information by using the martingale approach. Stettner (2004) studies the risk-sensitive portfolio maximization problem when the dynamics of the asset prices depend on some economical factors, which are completely or partially observed. Frey et al. (2012) solves the portfolio optimization problem under partial information by including expert opinions in the analysis as a second source of information. Concerning portfolio optimization problems under partial information we also refer to Pham (2011) which gives a very broad overview of previous studies in the subject.

The literature related with the large investor with price impact mainly considers direct impact on the underlying stock price process. In particular, Cvitanić et al. (1996), Cuoco and Cvitanić (1998) and Kraft and Kühn (2011) assume that the large investor has an influence on the drift and volatility of the price process via the dollar amount invested in the stock. Also, Ku and Zhang (2016) assume that the drift is affected by the speed of the investor's trading action. In this respect, our model deviates from the existing literature as we assume that the impact on the drift of the price process is a function of the fraction of wealth invested in the stock. Busch et al. (2013) has studied an optimal consumption and investment problem in which the price process follows a regime-switching jump-diffusion. By modeling the intensity of the regime switch as a function of the fraction of wealth invested in the stock, they allow the large investor to have an indirect, persistent effect. Note that there is also a growing literature on price impact models used in the context of optimal order execution problems where the stock price process is driven by a diffusion whose drift is a function of the volume or speed of trading (see, e.g., Almgren and Chriss 2001). For a detailed overview of price impact models in the context of optimal order execution, we refer to the surveys Gökay et al. (2011) and Gatheral and Schied (2013).

This paper is structured as follows. In Sect. 2, we introduce the modeling framework and the optimization problem. In Sect. 3, we solve the optimization problem under complete information for the case of logarithmic and power utility preferences and linear impact function. In Sect. 4, we introduce the partial information setting, derive the corresponding filtering equations and solve the optimization problem for the case of logarithmic and power utility and linear impact functions. Section 5 illustrates numerical results and Sect. 6 concludes.

2 Financial market model

We consider a finite time interval $[0, T]$ and a continuous-time finite-state Markov chain Y defined on the filtered probability space $(\Omega, \mathscr{G}, \mathbb{G}, \mathbb{P})$, where $\mathbb{G} = (\mathscr{G}_t)_{t\geq 0}$ satisfies the usual conditions; all processes we consider here are assumed to be $\mathbb{G}$-adapted. Y represents the uncertainty of the state of the market. We denote by $\mathscr{E} = \{e_1, e_2, ..., e_K\}$ the state space where, without loss of generality, we assume that e_k is the basis column vector of $\mathbb{R}^K$. Y has the generator $Q = (Q^{ij})$ and its initial distribution is denoted by $\Pi = (\Pi^1, \cdots, \Pi^K)$.

We have a large investor with given initial wealth $x \in \mathbb{R}^+$ and whose objective is to find self-financing investment strategies that maximize expected utility from terminal wealth. We consider a risk-free bond and a risky asset as the available instruments in the market. The bond price process has the dynamics

$$dB_t = rB_t dt, \ t \geq 0,$$

where $r > 0$ is the risk-free rate. Let $h_t \in \mathbb{R}$ denotes the fraction of the wealth that is invested in the risky asset at time t. Then, $1 - h_t$ denotes the fraction of the wealth invested in the bond at time t. In order to avoid technical difficulties we assume

A1 $h_t \in [-L, L]$, $L \in \mathbb{R}_+$ for all $t \in [0, T]$.

Note that in **A1**, L might be chosen large enough to guarantee that the optimal solution is an interior one.

The price of the risky asset evolves according to a diffusion whose drift is a function of the current state of the market and the fraction of the wealth invested in risky asset by the large investor. That is,

$$\frac{dS_t}{S_t} = (\mu(Y_t) + g(h_t))\,dt + \sigma dW_t, \quad S_0 = s, \tag{1}$$

where W is a $\mathbb{G}$-Brownian motion independent of Y and the function g represents the impact of the large investor on the drift of the price process. Also note that $\mu(Y_t) = MY_t$ with $M^k = \mu(e_k)$, $1 \le k \le K$. This is due to the finite-state property of the Markov chain. In the sequel we are going to use both notations interchangeably.

The dynamics in (1) suggests that the portfolio choice of the large investor might be taken as a signal by the rest of the market. That is, the portfolio choice of the large investor acts as a factor governing the drift of the stock price.

Throughout this paper we cover the case with no impact on volatility. Note that in this case the dynamics in (1) admits a unique weak solution provided that the function g is sufficiently regular. On the other hand, the case with an impact on volatility, i.e., with $\sigma(h_t)$, under partial information would bring us to an interesting setting where actions of the investor create a trade-off between the increase in the controlled part of the drift and the decrease in the precision of the estimates of the unobserved part of the drift. However, this setting is also technically more delicate and is left for future research.

The self-financing portfolio property assumption implies that the dynamics of the wealth of the investor satisfies

$$\frac{dX_t^{(h)}}{X_t^{(h)}} = h_t \frac{dS_t}{S_t} + (1 - h_t)\frac{dB_t}{B_t}.$$

That is,

$$\frac{dX_t^{(h)}}{X_t^{(h)}} = (h_t\,(\mu(Y_t) + g(h_t)) + (1 - h_t)r)\,dt + h_t \sigma dW_t, \tag{2}$$

$X_0^{(h)} > 0$. In order to ensure that the wealth process is well defined, we consider investment strategies that satisfy

A2 $\int_0^T \left(h_s X_s^{(h)}\right)^2 ds < \infty$ almost surely.

Recall that one of the assumptions in the classical setting is that the investors are price takers. In the current setting this assumption is violated as we allow the investor to have price impact. Hence one can not rely on the no-arbitrage condition provided for the classical setting. In this context, in the next theorem we derive the no-arbitrage condition on the impact function g.

Theorem 1 (No-arbitrage) *Let S be given by the SDE* (1)*,* **A1**, **A2** *are assumed to hold and the function g satisfies $|g(h_t)| \le C|h_t|$ for a positive constant C. Then, the market is arbitrage free.*

Proof For an admissible strategy h_t, let

$$\theta(t) = \frac{\mu(Y_t) - r}{\sigma}, \qquad 0 \le t \le T.$$

Since Y_t is adapted and σ is constant, $\theta(t)$ is a $\mathscr{G}_t$-adapted process. By Girsanov theorem, there is an equivalent probability measure $\tilde{\mathbb{P}}$ under which

$$\tilde{W}_t = W_t + \int_0^t \theta(s)\, ds$$

is a Brownian motion. We note that $|\theta(t)| \leq \frac{|\mu(Y_t)|+r}{\sigma}$ for $0 \leq t \leq T$ and the Novikov condition is satisfied. We also have the Radon–Nikodym derivative given by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(-\int_0^T \theta(t)\, dW_t - \frac{1}{2}\int_0^T \theta^2(t)\, dt\right).$$

Define, for any positive α,

$$L_t = -\exp\left(-\alpha R_t\right),$$

where

$$R_t = \int_0^t h_s\left(\mu(Y_s) + g(h_s) - r\right)\, ds + \int_0^t h_s \sigma\, dW_s.$$

Then we have

$$L_t = -\exp\left(-\alpha\left(\int_0^t h_s g(h_s)\, ds + \int_0^t h_s\sigma\, d\tilde{W}_s\right)\right).$$

By Itô's formula,

$$dL_t = L_t\left(\left(-\alpha h_t g(h_t) + \frac{1}{2}\alpha^2 h_t^2 \sigma^2\right) dt - \alpha h_t \sigma\, d\tilde{W}_t\right).$$

If $\alpha > \frac{2C}{\sigma^2}$, then the dt term becomes negative (note that L_t is negative). Considering the integrability condition on L_t we obtain L_t is a $\tilde{\mathbb{P}}$-supermartingale, thus

$$\tilde{\mathbb{E}}(L_T) \leq \tilde{\mathbb{E}}(L_0) = -1 \tag{3}$$

where $\tilde{\mathbb{E}}$ represents expectations under probability measure $\tilde{\mathbb{P}}$.

Let h_t be an admissible strategy that satisfies $\mathbb{P}\{e^{-rT}X_T^{(h)} \geq X_0^{(h)}\} = 1$, which is $\mathbb{P}\{R_T \geq 0\} = 1$. Since $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent, $\tilde{\mathbb{P}}\{R_T \geq 0\} = 1$ and using equation (3) we obtain $\tilde{\mathbb{P}}\{R_T = 0\} = 1$. This implies that $\mathbb{P}\{e^{-rT}X_T^{(h)} = X_0^{(h)}\} = 1$. Therefore, h_t cannot be an arbitrage strategy. $\square$

Suppose we are given a concave, increasing and twice continuously differentiable utility function $U : \mathbb{R}_+ \to \mathbb{R}$. Then, the problem of the large investor is to

$$\max_h \mathbb{E}^{x,i}[U(X_T^{(h)})],$$

subject to the initial value of the wealth $X_t^{(h)} = x$ and initial state $Y_t = i$.

3 Optimization problem under complete information

In this section, we assume that the investor's filtration is given by $\mathbb{G} = (\mathscr{G}_t)_{t\geq 0}$, that is, the investor is assumed to observe the true state of the market and the stock price. Accordingly, a portfolio strategy is *admissible* if $h_t \in \mathscr{G}_t$, for all t and **A1**, **A2** hold. We denote the set of admissible strategies by $\mathscr{H}$. In the following we solve the investment problems for the case of logarithmic and power utility preferences. The value function of the investor is denoted by

$$V(t, x, i) = \sup_{h\in\mathscr{H}} \mathbb{E}^{x,i}\left[\frac{1}{\theta}(X_T^{(h)})^\theta\right]. \tag{4}$$

3.1 Logarithmic utility

We first consider the portfolio optimization problem in the case of logarithmic utility. In this case it is possible to solve the optimization problem for a general impact function that is regular enough.

Proposition 1 *Suppose that g is continuously differentiable and $U(x) = \log(x)$. Then the optimal strategy*

$$h^* = \arg\max_{h\in\mathscr{H}} \mathbb{E}^{x,i}[U(X_T^{(h)})]$$

exists. Moreover, for all $(t, i) \in [0, T] \times \mathscr{E}$, $h^*(t, i) \in H_i^{log}$, where H_i^{log} is defined by

$$H_i^{log} = \{-L, L\} \cup \left\{l : M^i - r + g(l) + l\left(\frac{\partial g(l)}{\partial l} - \sigma^2\right) = 0\right\}. \tag{5}$$

Proof Given the dynamics in (2) we apply Itô's formula for $U(x) = \log(x)$ and obtain

$$U(X_T^{(h)}) = \log(x) + \int_t^T \left(h_s\left(\mu(Y_s) + g(h_s)\right) + (1 - h_s)r - \frac{h_s^2\sigma^2}{2}\right)ds + \int_t^T h_s\sigma dW_s.$$

For any $h \in \mathscr{H}$, the stochastic integral $\int_t^T h_s \sigma dW_s$ is well defined and has an expected value zero. Thus, we have

$$\mathbb{E}^{x,i}[U(X_T^{(h)})] = \log(x) + \mathbb{E}\left[\int_t^T \left(h_s\,(\mu(Y_s) + g(h_s)) + (1 - h_s)r - \frac{h_s^2\sigma^2}{2}\right) ds\right]. \tag{6}$$

Now we denote the integrand in (6) by

$$f(s,l) = l\,(\mu(Y_s) + g(l)) + (1 - l)r - \frac{l^2\sigma^2}{2}.$$

It follows from the continuity of g that for any $s \in [t, T]$, $f(s,\cdot)$ is a continuous function defined on the compact set $[-L, L]$. Hence, maximizer of $f(s,\cdot)$ exists. Moreover, the maximizer is an element of either $\{l : \frac{\partial f(s,l)}{\partial l} = 0\}$ or $\{-L, L\}$. That is, $h_s^* \in H_i^{log}$. □

Remark 1 It is possible to extend the result of Proposition 1 to the case where the function $g(\cdot)$ is differentiable except for finitely many points of the domain $[-L, L]$. Let H^0 denotes the set of the points where $g(\cdot)$ is not differentiable. Then, Proposition 1 holds with $h^*(t,i) \in \left(H_i^{log} \cup H^0\right)$.

As a specific case, we now consider the optimization problem under complete information with linear impact function and logarithmic utility. We set $g(h) = \beta h$, $\beta > 0$. The following corollary is an immediate result of Proposition 1.

Corollary 1 *Suppose $U(x) = \log(x)$ and $g(h) = \beta h$, $\beta > 0$. Then*

$$H_i^{log} = \left\{-L, L, \frac{M^i - r}{\sigma^2 - 2\beta}\right\}.$$

In particular, depending on the given set of model parameters we have the following cases:

i) if $2\beta - \sigma^2 < 0$, then, for all $(t,i) \in [0,T] \times \mathscr{E}$, the optimal strategy is given by

$$h^*(t,i) = \frac{M^i - r}{\sigma^2 - 2\beta},$$

and the value function has the following stochastic representation:

$$V(t,x,i) = log(x) + r(T-t) + \mathbb{E}^{x,i}\left[\int_t^T \frac{(\mu(Y_s) - r)^2}{2(\sigma^2 - 2\beta)} ds\right].$$

ii) If $2\beta - \sigma^2 \geq 0$, then, for all $(t,i) \in [0,T] \times \mathscr{E}$, the optimal strategy is given by

$$h^*(t,i) = L\left(\mathbb{1}_{\{M^i - r \geq 0\}} - \mathbb{1}_{\{M^i - r < 0\}}\right),$$

and the value function in this case has the following stochastic representation:

$$V(t,x,i)=log(x)+(T-t)\left(\left(\beta-\frac{\sigma^2}{2}\right)L^2+r\right)+\mathbb{E}^{x,i}\left[L\int_t^T|\mu(Y_s)-r|ds\right].$$

Remark 2 In Corollary 1 the case with parameter condition i) implies that if $M^i - r < 0$, which can be interpreted as an indication of an unfavorable market environment, optimal portfolio strategy is to put negative weight in the risky asset (short selling) and positive in the bank account and vice versa.

When the parameter condition ii) holds and if $M^i - r \geq 0$, then it is optimal to borrow as much as possible from the bank account and invest the proceeds in the risky asset. If, on the other hand, $M^i - r \leq 0$, then the optimal is to sell the risky asset short as much as possible and invest the proceeds in the bank account.

Remark 3 In the model without price impact the optimal portfolio strategy is given by (see, Bäuerle and Rieder 2004)

$$h^*(t,i)=\frac{M^i-r}{\sigma},$$

and the corresponding value function is

$$V(t,x,i)=log(x)+r(T-t)+\mathbb{E}^{x,i}\left[\int_t^T\frac{(\mu(Y_s)-r)^2}{2\sigma^2}ds\right]. \qquad (7)$$

Corollary 1 suggests that the value function of the investor in the presence of the price impact dominates the value function given in (7). That is, the investor benefits from the presence of the price impact.

3.2 Power utility

Next we assume that the utility function is $U(x)=\frac{1}{\theta}x^\theta$, $0<\theta<1$. This gives a constant relative risk aversion (CRRA) type preferences with risk aversion $(1-\theta)/x$. In contrast to the case of logarithmic utility it is not possible to solve the optimization problem directly. Instead, we address this problem by dynamic programing approach. To this, for any function $v \in C^{1,2}$ and $(t,x,i)\in[0,T]\times\mathbb{R}^+\times\mathscr{E}$, $h\in\mathscr{H}$, we define the differential operator

$$\begin{aligned}\mathscr{A}^h v(t,x,i)=&\frac{\partial v(t,x,i)}{\partial t}+\frac{\partial v(t,x,i)}{\partial x}x(h(M^i+g(h))+(1-h)r)\\&+\frac{1}{2}\frac{\partial^2 v(t,x,i)}{\partial x^2}x^2h^2\sigma^2+\sum_j(v(t,x,j)-v(t,x,i))Q^{ij}.\end{aligned}$$

Hypothetically, the following Hamilton–Jacobi–Bellman (HJB) equation has to be solved

$$\sup_h \mathscr{A}^h v(t,x,i) = 0$$
$$v(T,x,i) = \frac{1}{\theta}x^\theta \quad \text{for all} \quad (x,i) \in \mathbb{R}^+ \times \mathscr{E}. \tag{8}$$

Here note that due to the general form of the impact function g, it is not possible to characterize the solution for the current optimization problem. In the following we instead consider the case with a linear impact function and obtain explicit results.

Proposition 2 *Suppose $U(x) = \frac{1}{\theta}x^\theta$ and $g(h) = \beta h$, $\beta > 0$ and*

i) $2\beta - (1-\theta)\sigma^2 < 0$, then the optimal strategy h^* is given by

$$h^*(t,i) = \frac{M^i - r}{(1-\theta)\sigma^2 - 2\beta}, \tag{9}$$

and $V(t,x,i) = \frac{1}{\theta}x^\theta u(t,i)$, for all $(t,x,i) \in [0,T] \times \mathbb{R}^+ \times \mathscr{E}$, where $u(t,i) > 0$, with $u(T,i) = 1$, $i \in \mathscr{E}$, is the unique solution of the following system of linear differential equations

$$\frac{\partial u(t,i)}{\partial t} + a(i)u(t,i) + \sum_j (u(t,j) - u(t,i))Q^{ij} = 0, \tag{10}$$

with $a(i) = \theta r + \frac{\theta(M^i - r)^2}{2((1-\theta)\sigma^2 - 2\beta)}$. Moreover, the value function has the following stochastic representation

$$V(t,x,i) = \frac{x^\theta}{\theta}\exp(r\theta(T-t))\mathbb{E}^{x,i}\left[\exp\left(\int_t^T \frac{\theta(\mu(Y_s) - r)^2}{2((1-\theta)\sigma^2 - 2\beta)}ds\right)\right].$$

ii) $2\beta - (1-\theta)\sigma^2 \geq 0$, then the optimal strategy h^* is given by

$$h^*(t,i) = L\left(1_{\{M^i - r \geq 0\}} - 1_{\{M^i - r < 0\}}\right),$$

and $V(t,x,i) = \frac{1}{\theta}x^\theta u(t,i)$, for all $(t,x,i) \in [0,T] \times \mathbb{R}^+ \times \mathscr{E}$, where $u(t,i) > 0$, with $u(T,i) = 1$, $i \in \mathscr{E}$, is the unique solution of the following system of linear differential equations

$$\frac{\partial u(t,i)}{\partial t} + a(i)u(t,i) + \sum_j (u(t,j) - u(t,i))Q^{ij} = 0, \tag{11}$$

with $a(i) = \theta r + \theta L|M^i - r| + \theta L^2\left(\beta + \frac{(\theta-1)\sigma^2}{2}\right)$. Moreover, the value function has the following stochastic representation

$$V(t,x,i) = \frac{x^\theta}{\theta}\mathbb{E}^{x,i}\left[\exp\left(\theta(T-t)\left(L^2\left(\beta - \frac{(1-\theta)\sigma^2}{2}\right) + r\right)\right.\right.$$
$$\left.\left. + \theta L \int_t^T |\mu(Y_s) - r| ds\right)\right].$$

Proof It follows from the form of the utility function and the linear structure of the dynamics of the wealth process that for all $i \in \{1, \dots, K\}$ the value function can be rewritten as $v(t,x,i) = \frac{1}{\theta}x^\theta u(t,i)$, for some function $u \geq 0$ with $u(T,i) = 1$. This gives following partial derivatives

$$\frac{\partial v(t,x,i)}{\partial t} = \frac{1}{\theta}x^\theta \frac{\partial u(t,i)}{\partial t}, \quad \frac{\partial v(t,x,i)}{\partial x} = x^{\theta-1}u(t,i),$$
$$\frac{\partial^2 v(t,x,i)}{\partial x^2} = (\theta-1)x^{\theta-2}u(t,i).$$

Substituting these and $g(h) = \beta h$ in (8), we obtain

$$-ru(t,i) = \sup_{h\in[-L,L]}\left\{h(M^i - r)u(t,i) + h^2(\beta + \frac{\theta-1}{2}\sigma^2)u(t,i)\right\}$$
$$+ \frac{1}{\theta}\frac{\partial u(t,i)}{\partial t} + \frac{1}{\theta}\sum_j Q^{ij}(u(t,j) - u(t,i)),$$
$$u(T,i) = 1 \quad \text{for all} \quad i \in \{1, \dots, K\}. \tag{12}$$

We have the following necessary condition for the maximizer

$$2h(\beta + \frac{\sigma^2(\theta-1)}{2})u(t,i) + (M^i - r)u(t,i) = 0.$$

Suppose $2\beta < (1-\theta)\sigma^2$. These together with $u(t,i) > 0$ (for the positivity, see Remark 4 below) imply that the necessary condition is also sufficient. That is, the maximizer is given by (9). Inserting this maximum and after some simple algebra, we obtain (10). This differential equation has a unique solution u and we have the following Feynman–Kac type representation of $u(t,i)$ (see Bäuerle and Rieder 2004, Lemma 2)

$$u(t,x,i) = \exp(r\theta(T-t))\mathbb{E}^{x,i}\left[\exp\left(\int_t^T \frac{\theta(\mu(Y_s)-r)^2}{2((1-\theta)\sigma^2 - 2\beta)}ds\right)\right].$$

In fact, this function $v(t,x,i) = \frac{1}{\theta}x^\theta u(t,i)$ is a solution of the HJB equation (8), $v \in C^{1,2}$, and satisfies $|v(t,x,i)| \leq K(1+|x|)$ for a suitable constant K. By applying a Verification Theorem (see, e.g., Bäuerle and Rieder 2004, Theorem 1), we obtain $v(t,x,i)$ is indeed an optimal value function $V(t,x,i)$.

Next suppose $2\beta \geq (1-\theta)\sigma^2$. Then, the maximum is attained in one of the end points of the interval $[-L, L]$. It is clear by inspection that the maximizer depends on the value of $(M^i - r)$. Namely, $h^*(t,i) = L$ for $M^i - r > 0$ and $h^*(t,i) = -L$ otherwise. Hence, inserting these in (12) and after some simple algebra we obtain the system of differential equations in (11). By the same argument as in case i), the proof is completed. □

Remark 4 Note that the above representations of u imply that the function $u(t,i)$ stays positive provided that the given parameter restrictions are satisfied.

Remark 5 Proposition 2 suggests that for any parameter condition the current value function dominates the value function given in Bäuerle and Rieder (2004), Theorem 3. This means that the investor benefits from the presence of the price impact also in the case of power utility preferences.

4 Optimal control under partial information

Throughout this section we assume that the state process Y is not directly observable by the large investor. Instead, she observes the price process S and knows the model parameters, that is, Π, Q, M and the function $g(\cdot)$. Hence, information available to the large investor is carried by the filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$, $\mathscr{F}_t = \sigma\{S_u, 0 \leq u \leq t\}$. We note that $\mathscr{F}_t \subset \mathscr{G}_t$.

Recall that the optimization problem of the large investor is to find investment strategies that maximize the expected utility from terminal wealth. We assume that this decision depends only on the information available to the investor at time t. That is, we consider the self-financing investment strategies h such that h_t is $\mathscr{F}_t$-adapted.

Accordingly, an $\mathbb{F}$-adapted self-financing investment strategy which satisfy **A1** and **A2** is called an *admissible investment strategy*. We denote the set of admissible investment strategies by $\mathscr{H}^{\mathscr{F}}$.

Suppose we are given a concave, increasing and twice continuously differentiable utility function $U : \mathbb{R}_+ \to \mathbb{R}$. The optimization problem of the large investor is given by

$$\sup_{h\in\mathscr{H}^{\mathscr{F}}} \mathbb{E}^x \left[U(X_T^{(h)}) \right],$$

where $\mathbb{E}^x$ denotes the conditional expectation given $X_0 = x$.

Considering $\mathbb{F}$-adapted investment strategies, we naturally end up with an optimal control problem under partial information. In the next part, in order to solve this problem we will derive an equivalent control problem under complete information via the so-called reduction approach (see, e.g., Fleming and Pardoux 1982).

4.1 Reduction of the optimal control problem

The reduction approach requires the derivation of the filtering equation for the unobservable state of the underlying state variable. In what follows we will denote the filter for the unobserved state of the Markov chain by $p_t = (p_t^1, \cdots, p_t^K)$ with

$p_t^k = \mathbb{P}(Y_t = e_k|\mathscr{F}_t)$. It follows from the finite-state property of the Markov chain that

$$\mathbb{E}[\mu(Y_t)|\mathscr{F}_t] = \sum_{k=1}^{K} \mu(e_k) p_t^k = M p_t.$$

We now introduce the process

$$\hat{W}_t := W_t + \int_0^t \frac{\mu(Y_s) - M p_s}{\sigma} ds. \tag{13}$$

The following lemma shows that $\hat{W}_t$ is an $\mathbb{F}$-Brownian motion and gives the $\mathscr{F}_t$-dynamics of the filter process p.

Lemma 1 *The process given in (13) is an $\mathscr{F}_t$-Brownian motion. Moreover, the filter process p_t^k, $1 \le k \le K$ is the unique solution of*

$$dp_t^k = \sum_j Q^{jk} p_t^j dt + \left(\frac{M^k - M p_t}{\sigma}\right) p_t^k d\hat{W}_t, \tag{14}$$

with the initial condition $p_0^k = \Pi^k$.

Proof In order to prove that $\hat{W}$ is an $\mathbb{F}$-Brownian motion we will follow the similar arguments given in the proof of Elliott (1982, Lemma 1.4). First, it follows from definition (13) and the Fubini theorem that we have

$$\begin{aligned}\mathbb{E}\left[\hat{W}_t - \hat{W}_s \middle| \mathscr{F}_s\right] &= \mathbb{E}\left[\int_s^t \frac{\mu(Y_u) - M p_u}{\sigma} du + W_t - W_s \middle| \mathscr{F}_s\right] \\ &= 0 \text{ a.s..}\end{aligned}$$

Hence, $\hat{W}$ is an $\mathbb{F}$-martingale. Second, $\hat{W}$ is a continuous $\mathbb{G}$-semimartingale with the quadratic variation $\langle \hat{W}, \hat{W} \rangle_t = \langle W, W \rangle_t = t$. As this quadratic variation process is deterministic, it stays same under a change of filtration. That is, the $\mathbb{F}$-quadratic variation of $\hat{W}$ at time t is also equal to t. Finally, it follows from the Levy's characterization of the Brownian motion that $\hat{W}$ is an $\mathbb{F}$-Brownian motion.

Showing that $\hat{W}$ is a Brownian motion with respect to the observation filtration $\mathbb{F}$, we can make use of the well-known martingale representation results. This brings us to a situation where we can apply the standard results given in, for example, Wonham (1964) and Elliott et al. (1994, Chapter 8), and obtain (14). □

Now it follows from (2) and Lemma 1 that $\mathbb{F}$ semimartingale decomposition of X is given by

$$\frac{dX_t^{(h)}}{X_t^{(h)}} = (h_t (M p_t + g(h_t)) + (1 - h_t) r)\, dt + h_t \sigma d\hat{W}_t, \quad X_0 = x. \tag{15}$$

Then, it follows from (14) and (15) that the $(K+1)$-dimensional process $(X, p) \in [0, T] \times \mathbb{R}^{+} \times \Delta_K$, where Δ_K is the K-dimensional simplex, is an $\mathbb{F}$-Markov process. Considering this $(K+1)$-dimensional process as the state process, we introduce the equivalent optimal control problem under complete information with the following:

$$\max_{h} \mathbb{E}^{x,\mathbf{p}}[U(X_T^{(h)})],$$

where $\mathbb{E}^{x,\mathbf{p}}$ denotes the conditional expectation given $X_0 = x$ and $p_0 = p$. Accordingly, the value function of the investor in the reduced model is denoted by the following

$$V(t, x, \mathbf{p}) = \sup_{h \in \mathscr{H}^{\mathscr{F}}} \mathbb{E}^{x,\mathbf{p}}[U(X_T^{(h)})]. \tag{16}$$

4.2 Logarithmic utility

As the first case we consider the portfolio optimization problem for a large investor with logarithmic utility preferences, that is, we assume that $U(x) = \log(x)$. In this case, the optimal control problem can be solved directly.

Proposition 3 *Suppose g is continuously differentiable and $U(x) = \log(x)$. Then the optimal strategy*

$$h^* = \arg_{h \in \mathscr{H}^{\mathscr{F}}} \max \mathbb{E}^{x,\mathbf{p}}[U(X_T^{(h)})]$$

exists. Moreover, for all $t \in [0, T]$, $h_t^* \in H_t^{log}$, where H_t^{log} is defined by

$$H_t^{log} = \{-L, L\} \cup \left\{ l : Mp_t - r + g(l) + l(\frac{\partial g(l)}{\partial l} - \sigma^2) = 0 \right\}. \tag{17}$$

Proof Given the dynamics in (15) we apply Itô's formula for $U(X_t) = \log(X_t)$ and get

$$U(X_T^{(h)}) = \log(x) + \int_t^T (h_s (Mp_s + g(h_s)) \\ +(1 - h_s)r - \frac{h_s^2 \sigma^2}{2} \Bigg) ds + \int_t^T h_s \sigma d\hat{W}_s.$$

For any $h \in \mathscr{H}^{\mathscr{F}}$, the stochastic integral $\int_t^T h_s \sigma d\hat{W}_s$ is well defined and has an expected value zero. Thus, we have

$$\mathbb{E}^{x,\mathbf{p}}[U(X_T^{(h)})] = \log(x) + \mathbb{E}^{x,\mathbf{p}} \left[\int_t^T \left(h_s (Mp_s + g(h_s)) + (1 - h_s)r - \frac{h_s^2 \sigma^2}{2} \right) ds \right]. \tag{18}$$

Now we denote the integrand in (18) by

$$f(s,l) = l\,(Mp_s + g(l)) + (1-l)r - \frac{l^2\sigma^2}{2}.$$

It follows from the continuity of g that for any $s \in [t, T]$, $f(s,\cdot)$ is a continuous function defined on the compact set $[-L, L]$. Hence, a maximizer of $f(s,\cdot)$ exists. Moreover, the maximizer is an element of either $\left\{l : \frac{\partial f(s,l)}{\partial l} = 0\right\}$ or $\{-L, L\}$. That is, $h_s^* \in H_s^{log}$ for all $s \in [t, T]$. □

Remark 6 It is possible to extend the result of Proposition 3 to the case where the functions $g(\cdot)$ is continuously differentiable except for finitely many points of the domain $[-L, L]$. Let H^0 denotes the set of the points where $g(\cdot)$ is not differentiable. Then, Proposition 3 holds with $h_t^* \in \left(H_t^{log} \cup H^0\right)$.

Next, as an example we consider the optimization problem under partial-information with linear impact function and logarithmic utility. Formally, we assume $g(h) = \beta h$, $\beta > 0$. The following corollary follows from Proposition 3.

Corollary 2 *Suppose $U(x) = \log(x)$ and $g(h) = \beta h$, $\beta > 0$. Then, for all $t \in [0, T]$,*

$$H_t^{log} = \left\{-L, L, \frac{Mp_t - r}{\sigma^2 - 2\beta}\right\}.$$

In particular, depending on the given set of model parameters we have the following cases:

i) If $2\beta - \sigma^2 < 0$, then, for all $t \in [0, T]$, the optimal portfolio strategy is given by

$$h_t^* = \frac{Mp_t - r}{\sigma^2 - 2\beta},$$

and the value function $V(t, x, \mathbf{p})$, for all $(t, x, \mathbf{p}) \in [0, T] \times \mathbb{R}^+ \times \Delta_K$, has the following stochastic representation:

$$V(t, x, \mathbf{p}) = log(x) + r(T-t) + \mathbb{E}^{x,\mathbf{p}}\left[\int_t^T \frac{(Mp_s - r)^2}{2(\sigma^2 - 2\beta)} ds.\right]$$

ii) If $2\beta - \sigma^2 \geq 0$, then, for all $t \in [0, T]$, the optimal portfolio strategy is given by

$$h_t^* = L\left(\mathbb{1}_{\{Mp_t - r \geq 0\}} - \mathbb{1}_{\{Mp_t - r < 0\}}\right),$$

and the value function $V(t, x, \mathbf{p})$, for all $(t, x, \mathbf{p}) \in [0, T] \times \mathbb{R}^+ \times \Delta_K$, has the following stochastic representation:

$$V(t, x, \mathbf{p}) = log(x) + (T-t)\left((\beta - \frac{\sigma^2}{2})L^2 + r\right) + \mathbb{E}^{x,i}\left[L\int_t^T |Mp_s - r| ds\right].$$

Remark 7 Here, if $Mp_t - r < 0$ (that is, if the current market environment is inferred to be unfavorable) the optimal strategy is to put negative weight in the risky asset (short selling) and to invest proceeds in the bank account. On the other hand, if $Mp_t - r \geq 0$, then it is optimal to borrow as much as possible from the riskless rate and invest the amount in the risky asset.

In all of the above cases, when compared to the case with complete information the resulting optimal strategy is obtained by replacing the unknown drift by its filter estimate. That is, the certainty equivalence principle holds.

4.3 Power utility

In what follows we assume that $U(x) = \frac{1}{\theta}x^\theta$, $0 < \theta < 1$. Here we mainly follow the procedure given in the previous section and solve the problem by using dynamic programming methods. To begin with, for any function $v \in C^{1,2}$ and $(t, x, \mathbf{p}) \in [0, T] \times \mathbb{R}^+ \times \Delta_K$, $h \in \mathscr{H}^{\mathscr{F}}$ we define the differential operator

$$\begin{aligned}\mathscr{A}^h v(t,x,\mathbf{p}) = {} & \frac{\partial v(t,x,\mathbf{p})}{\partial t} + \frac{\partial v(t,x,\mathbf{p})}{\partial x} x(h(M\mathbf{p} + g(h)) + (1-h)r) \\ & + \frac{1}{2}\frac{\partial^2 v(t,x,\mathbf{p})}{\partial x^2} x^2 h^2 \sigma^2 + \sum_{k,j} \frac{\partial v(t,x,\mathbf{p})}{\partial p_k} Q^{jk} p_j \\ & + \frac{1}{2\sigma^2} \sum_{k,j} \frac{\partial^2 v(t,x,\mathbf{p})}{\partial p_k \partial p_j} (M^k - M\mathbf{p})(M^j - M\mathbf{p}) p_k p_j \\ & + xh \sum_k \frac{\partial^2 v(t,x,\mathbf{p})}{\partial x \partial p_k} (M^k - Mp_t) p_k. \end{aligned} \tag{19}$$

Hypothetically, the value function solves the following HJB equation

$$\begin{aligned} & \sup_h \mathscr{A}^h v(t,x,\mathbf{p}) = 0 \\ & v(T,x,\mathbf{p}) = \frac{1}{\theta} x^\theta \quad \text{for all} \quad (x,\mathbf{p}) \in \mathbb{R}^+ \times \Delta_K. \end{aligned} \tag{20}$$

Due to the general form of the impact function g, it is not possible to show the existence of a solution or to characterize it as the solution of equation (20) for the current optimization problem. Instead we consider the optimization problem under partial information with power utility preferences and linear impact function. This case allows to derive the optimal control in the feedback form and yields a probabilistic representation for the corresponding value function.

Proposition 4 *Suppose* $U(x) = \frac{1}{\theta}x^\theta$, $g(h) = \beta h$, $\beta > 0$ *and* $2\beta - (1-\theta)\sigma^2 < 0$. *Define* $\gamma = \frac{(1-\theta)\sigma^2 - 2\beta}{\sigma^2 - 2\beta}$. *The value function is given by* $V(t,x,\mathbf{p}) = \frac{1}{\theta}x^\theta u(t,\mathbf{p})^\gamma$, *for all* $(t,x,\mathbf{p}) \in [0,T] \times \mathbb{R}^+ \times \Delta_K$, *where* $u > 0$, *with* $u(T,\mathbf{p}) = 1$, *for all* $\mathbf{p} \in \Delta_K$, *is the solution of the parabolic partial differential equation*

$$u(t,\mathbf{p})\left(\frac{\theta}{\gamma}\left(r+\frac{(M\mathbf{p}-r)^2}{2\gamma(\sigma^2-2\beta)}\right)\right)+\frac{1}{2\sigma^2}\sum_{k,j}\frac{\partial^2 u(t,\mathbf{p})}{\partial p_k\partial p_j}(M^k-M\mathbf{p})(M^j-M\mathbf{p})p_kp_j$$
$$+\sum_k\frac{\partial u(t,\mathbf{p})}{\partial p_k}\left(\sum_j Q^{jk}p_j+\frac{\theta}{\gamma}\frac{(M\mathbf{p}-r)(M^k-M\mathbf{p})}{\sigma^2-2\beta}p_k\right)+\frac{\partial u(t,\mathbf{p})}{\partial t}=0. \tag{21}$$

Moreover, the optimal portfolio strategy h^* is given in feedback form as

$$h^*(t,\mathbf{p})=\frac{M\mathbf{p}-r}{(1-\theta)\sigma^2-2\beta}+\frac{\gamma\sum_k\frac{\partial u(t,\mathbf{p})}{\partial p_k}(M^k-M\mathbf{p})p_k}{u(t,\mathbf{p})((1-\theta)\sigma^2-2\beta)}. \tag{22}$$

Proof Due to the form of the utility function and the linear structure of the dynamics of the wealth process, we use the ansatz $v(t,x,\mathbf{p})=\frac{1}{\theta}x^\theta u(t,\mathbf{p})^\gamma$, for some function $u^\gamma>0$ with $u(T,\mathbf{p})=1$, for all $\mathbf{p}\in\Delta_K$. This gives the following partial derivatives

$$\frac{\partial v(t,x,\mathbf{p})}{\partial t}=\frac{1}{\theta}x^\theta\gamma u(t,\mathbf{p})^{\gamma-1}\frac{\partial u(t,\mathbf{p})}{\partial t},\quad \frac{\partial v(t,x,\mathbf{p})}{\partial x}=x^{\theta-1}u(t,\mathbf{p})^\gamma,$$
$$\frac{\partial^2 v(t,x,\mathbf{p})}{\partial x^2}=(\theta-1)x^{\theta-2}u(t,\mathbf{p})^\gamma,\quad \frac{\partial v(t,x,\mathbf{p})}{\partial p_k}=\frac{1}{\theta}x^\theta\gamma u(t,\mathbf{p})^{\gamma-1}\frac{\partial u(t,\mathbf{p})}{\partial p_k},$$
$$\frac{\partial^2 v(t,x,\mathbf{p})}{\partial p_k\partial p_j}=\frac{1}{\theta}x^\theta\left(\gamma(\gamma-1)u(t,\mathbf{p})^{\gamma-2}\frac{\partial u}{\partial p_k}\frac{\partial u}{\partial p_j}+\gamma u(t,\mathbf{p})^{\gamma-1}\frac{\partial^2 u(t,\mathbf{p})}{\partial p_k\partial p_j}\right),$$
$$\frac{\partial^2 v(t,x,\mathbf{p})}{\partial p_k\partial x}=x^{\theta-1}\gamma u(t,\mathbf{p})^{\gamma-1}\frac{\partial u(t,\mathbf{p})}{\partial p_k}.$$

Substituting these and $g(h)=\beta h$ in (20), we obtain the following equation for function u for some γ that will be determined below.

$$\sup_{h\in[-L,L]}\Big\{u(t,\mathbf{p})(h(M\mathbf{p}+\beta h)+(1-h)r)+\frac{\theta-1}{2}u(t,\mathbf{p})h^2\sigma^2$$
$$+h\gamma\sum_k\frac{\partial u(t,\mathbf{p})}{\partial p_k}(M^k-M\mathbf{p})p_k\Big\}+\frac{1}{2\sigma^2}\frac{\gamma}{\theta}\sum_{k,j}(M^k-M\mathbf{p})(M^j-M\mathbf{p})p_kp_j$$
$$\times\left(\frac{\partial^2 u(t,\mathbf{p})}{\partial p_k\partial p_j}+(\gamma-1)u(t,\mathbf{p})^{-1}\frac{\partial u(t,\mathbf{p})}{\partial p_k}\frac{\partial u(t,\mathbf{p})}{\partial p_j}\right)+\frac{\gamma}{\theta}\frac{\partial u(t,\mathbf{p})}{\partial t}$$
$$+\frac{\gamma}{\theta}\sum_{k,j}\frac{\partial u(t,\mathbf{p})}{\partial p_k}Q^{jk}p_j=0, \tag{23}$$

with $u(T,\mathbf{p})=1$. The necessary condition for the optimizer of the above equation is given by

$$h(2\beta-(1-\theta)\sigma^2)u(t,\mathbf{p})+(M\mathbf{p}-r)u(t,\mathbf{p})+\gamma\sum_k\frac{\partial u(t,\mathbf{p})}{\partial p_k}(M^k-M\mathbf{p})p_k=0.$$

Provided that $2\beta - (1-\theta)\sigma^2 < 0$ and $u(t,\mathbf{p}) > 0$ (see Proposition 5 below) the necessary condition is also sufficient. This suggests that the maximizer is (22). Now we multiply (23) by θ/γ and insert h^*. This gives

$$\begin{aligned}0 &= \frac{\partial u(t,\mathbf{p})}{\partial t} + \sum_{k,j} \frac{\partial u(t,\mathbf{p})}{\partial p_k} Q^{jk} p_j + \frac{\theta}{\gamma} u(t,\mathbf{p}) r \\ &\quad + h^*(t,\mathbf{p})^2 u(t,\mathbf{p}) \frac{\theta}{\gamma}\left(\frac{(1-\theta)\sigma^2}{2} - \beta\right) + \frac{1}{2\sigma^2}\sum_{k,j}(M^k - M\mathbf{p})(M^j - M\mathbf{p}) p_k p_j \\ &\quad \times \left(\frac{\partial^2 u(t,\mathbf{p})}{\partial p_k \partial p_j} + (\gamma-1) u(t,\mathbf{p})^{-1} \frac{\partial u(t,\mathbf{p})}{\partial p_k}\frac{\partial u(t,\mathbf{p})}{\partial p_j}\right). \end{aligned} \tag{24}$$

This is a non-linear equation. In what follows, in order to eliminate the non-linearity in (24) we follow the idea given in Zariphopoulou (2001) and choose γ to satisfy

$$\gamma = \frac{(1-\theta)\sigma^2 - 2\beta}{\sigma^2 - 2\beta}.$$

With this choice of γ we have

$$\begin{aligned}0 &= \left(\frac{\gamma \sum_k \frac{\partial u}{\partial p_k}(M^k - M\mathbf{p}) p_k}{u(t,\mathbf{p})((1-\theta)\sigma^2 - 2\beta)}\right)^2 u(t,\mathbf{p}) \frac{\theta}{\gamma}\left(\frac{(1-\theta)\sigma^2}{2} - \beta\right) \\ &\quad + \left(\gamma \sum_k \frac{\partial u}{\partial p_k}(M^k - M\mathbf{p}) p_k\right)^2 \frac{1}{u(t,\mathbf{p})} \frac{\gamma - 1}{2\sigma^2},\end{aligned}$$

and hence we get the linear parabolic differential equation given in (21). Here note that the coefficients are continuous and bounded functions and hence there exists a solution to this parabolic differential equation, and also the solution to this Cauchy problem is unique (see, for example, Friedman 1983).

We have proved that the function $v(t,x,\mathbf{p}) = \frac{1}{\theta} x^\theta u(t,\mathbf{p})^\gamma$ is a solution of the HJB equation (20). Next we will show $v(t,x,\mathbf{p})$ is indeed the optimal value function $V(t,x,\mathbf{p})$. Let $h \in \mathscr{H}^{\mathscr{F}}$ be an arbitrary investment strategy. By applying Itô's formula to $v(t,x,\mathbf{p})$, we have

$$\begin{aligned} v(T, X_T^{(h)}, p_T) &= v(t,x,\mathbf{p}) + \int_t^T \mathscr{A}^h v(s, X_s^{(h)}, p_s)\, ds \\ &\quad + \int_t^T \frac{\partial v(s, X_s^{(h)}, p_s)}{\partial x} X_s^{(h)} h_s \sigma\, d\hat{W}_s \\ &\quad + \int_t^T \sum_k \frac{\partial v(s, X_s^{(h)}, p_s)}{\partial p_k}\left(\frac{M^k - M p_s}{\sigma}\right) p_s^k\, d\hat{W}_s \end{aligned}$$

$$\leq v(t,x,\mathbf{p}) + \int_t^T (X_s^{(h)})^\theta u(s,p_s)^\gamma h_s \sigma \, d\hat{W}_s$$

$$+ \int_t^T \sum_k \frac{1}{\theta}(X_s^{(h)})^\theta \gamma u(s,p_s)^{\gamma-1} \frac{\partial u(s,p_s)}{\partial p_k} \left(\frac{M^k - Mp_s}{\sigma} \right) p_s^k \, d\hat{W}_s \tag{25}$$

Note that the inequality (25) follows from the HJB equation. Under **A2**, the above stochastic integrals with respect to $\hat{W}_s$ are local martingales. Considering they are bounded below due to the fact $v \geq 0$, they are supermartingales. Taking conditional expectations,

$$\mathbb{E}^{t,x,\mathbf{p}}\left[\frac{1}{\theta}(X_T^{(h)})^\theta\right] \leq v(t,x,\mathbf{p}).$$

Therefore, we have $V(t,x,\mathbf{p}) \leq v(t,x,\mathbf{p})$. If we have a portfolio strategy $h^*(t,\mathbf{p})$ given by (22), we have equality in (25) with $h^*(t,\mathbf{p})$. Moreover, $h^*(t,\mathbf{p})$ is bounded, $u(s,p_s)$ and $\frac{\partial u(s,p_s)}{\partial p_k}$ are continuous, thus the stochastic integrals are in fact martingales. Taking conditional expections, we have

$$\mathbb{E}^{t,x,\mathbf{p}}\left[\frac{1}{\theta}(X_T^{(h^*)})^\theta\right] = v(t,x,\mathbf{p}).$$

The proof is now completed. □

Remark 8 If $2\beta - (1-\theta)\sigma^2 > 0$, the optimal strategy will be either L or $-L$. For the complete information case, this decision only depends the relation between the Markov modulated part of the drift and the risk-free rate. On the contrary, for the partial information case the decision is rather complicated and depends on the values of the variables in (23).

Here, comparing the resulting optimal control with the control corresponding to the complete information case, we conclude that the certainty equivalence principle does not hold for the case of power utility. In particular, in the current case there is an additional term arising due to the uncertainty about the state of the market.

Proposition 5 *Suppose* $2\beta-(1-\theta)\sigma^2 < 0$ *holds. Then, for all* $(t,\mathbf{p}) \in [0,T]\times\Delta_K$*, the function* u *has the following stochastic representation,*

$$u(t,\mathbf{p}) = \mathbb{E}^{\mathbb{P}^*}\left[\exp\left\{\frac{\theta r}{\gamma}(T-t) + \frac{\theta}{2\gamma^2(\sigma^2-2\beta)}\int_t^T (Mp_s - r)^2 ds\right\} \Big| p_t = \mathbf{p}\right], \tag{26}$$

where the kth *component of process* p *has the following dynamics under measure* $\mathbb{P}^*$*:*

$$dp_t^k = \left(\sum_j Q^{jk} p_t^j + \frac{\theta}{\gamma}\frac{(Mp_t - r)(M^k - Mp_t)}{\sigma^2 - 2\beta} p_t^k\right) dt + \frac{M^k - Mp_t}{\sigma} p_t^k dW_t^{\mathbb{P}^*}.$$

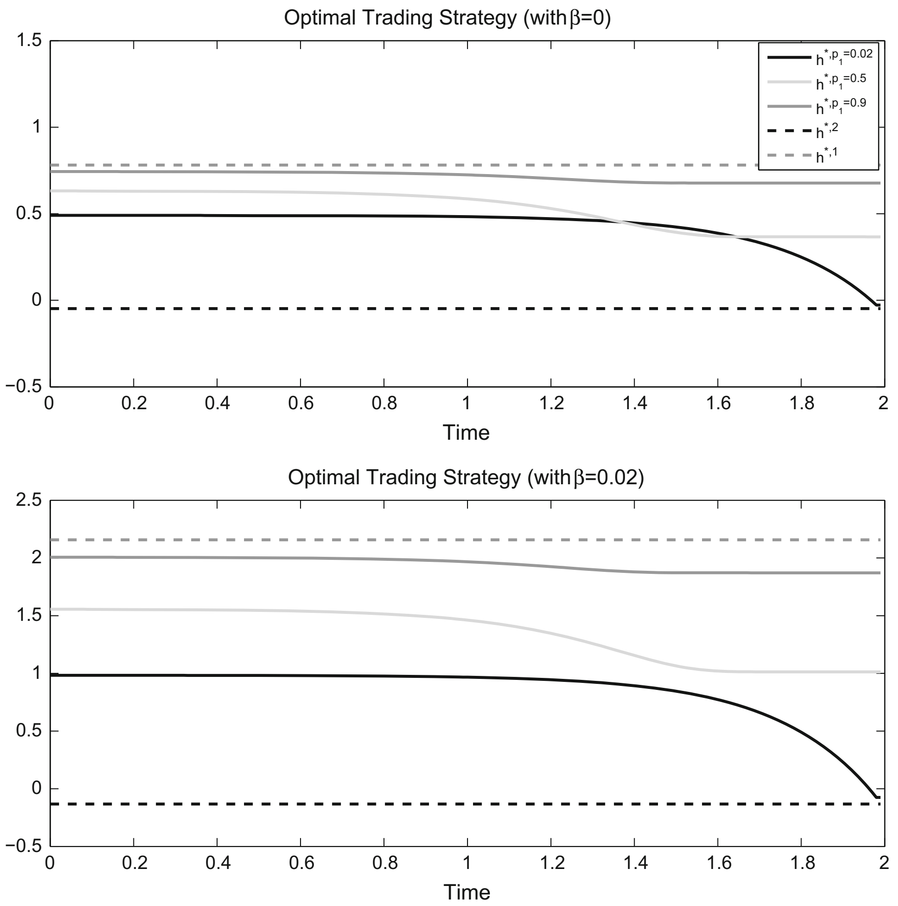

Fig. 1 Optimal portfolio strategy for power utility preferences with and without price impact for full (*dashed*) and partial (*solid*) information settings. $h^{*,1}$ ($h^{*,2}$): optimal strategy when the Markov chain is in the good (bad) state, $h^{*,p_1=a}$: optimal strategy when the filter is in state $\mathbf{p} = (a, 1-a)$

Proof Result immediately follows from the application of the Feynman-Kac theorem (see, e.g., Pham (2009, Thm 1.3.17) for the pde given in (21). □

5 Numerical study

In the numerical study, we first compare the optimal trading strategy for the full and partial information settings with and without price impact. To this, we set the following set of parameters: $T = 2$, $\beta = 0.02$, $\sigma = 0.28$, $M = (0.06, -0.002)^\top$, $r = 0.001$, $\theta = 0.3$, $(Q^{12}, Q^{21}) = (2, 0.01)$. Optimal trading strategies for the full-information case obtained in a straightforward way, that is, by inserting the parameters on the formulas given in Proposition 2. On the other hand, to obtain the corresponding optimal strategies in the case of partial information, we use an explicit finite-difference

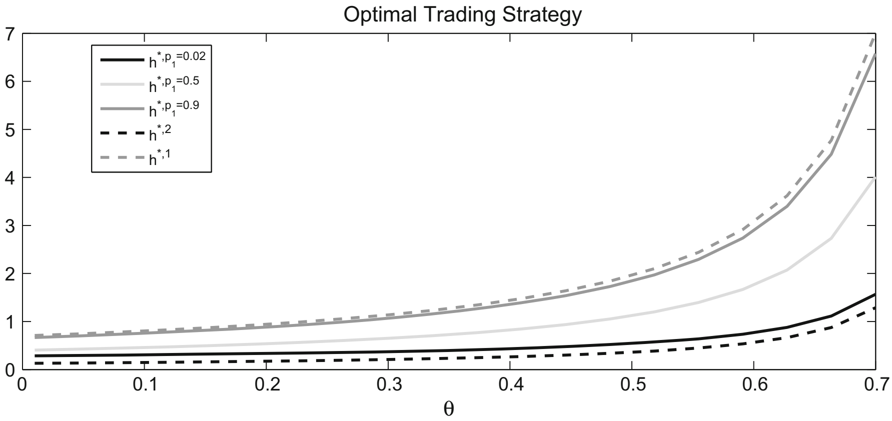

Fig. 2 Optimal portfolio strategy with full (*dashed*) and with partial (*solid*) information for power utility preferences with different levels of risk aversion. $h^{*,1}$ ($h^{*,2}$): optimal strategy when the Markov chain is in the good (bad) state, $h^{*,p_1=a}$: optimal strategy when the filter is in state $\mathbf{p} = (a, 1 - a)$

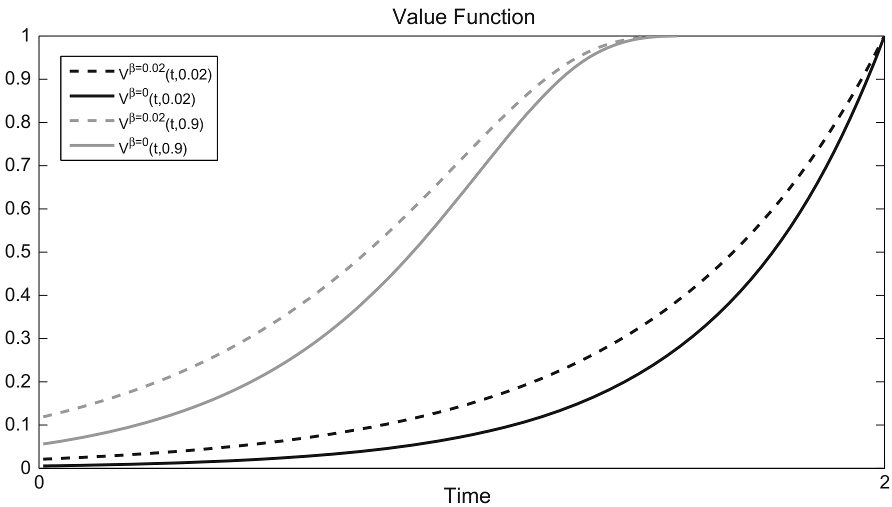

Fig. 3 Value function for partial information case with (*dashed*) and without (*solid*) price impact for power utility preferences when the filter is in state $\mathbf{p} = (0.02, 0.98)$ and $\mathbf{p} = (0.9, 0.1)$

method and solve the pde given in (21) numerically. Figure 1 shows that the presence of the price impact yields an upward shift on the optimal trading strategies, with an amount depending on the time and on the state of the filter. In particular, the higher the value of $\mathbf{p}^1$, the larger the shift. Note also that the amount of the shift decreases as the time gets closer to the maturity time T.

Next, we analyze the behavior of optimal trading strategies with respect to the risk aversion level, represented by θ. Figure 2 suggests that all types of optimal trading strategies are increasing with a decreasing level of risk aversion.

We finally compare the value functions of the optimal control problem under partial information with and without price impact. We conclude that the value function with price impact dominates the one without the price impact for any state of the filter. Results for the case when the filter is in state $\mathbf{p} = (0.02, 0.98)$ and $\mathbf{p} = (0.9, 0.1)$ are given in Fig. 3.

6 Conclusion

We investigate the problem of maximizing the expected utility from terminal wealth of a large investor whose action has some lasting impact on the price process under the changing market environment. We solve the optimal investment problem explicitly with the linear price impact under complete information. The decision of optimal strategies depends on the relationship between the Markov modulated part of the drift and the risk-free rate, and the relationship between the price impact part of the drift and the volatility. We then study the investment problem further for the large investor under partial information, and obtain the optimal investment strategies by the reduction approach. For power utility, the optimal strategy is given, unlike the complete information case, with an additional term due to the uncertainty about the market condition. We observe, for logarithmic and power utility functions treated in the paper, the large investor would gain benefits from the price impact by choosing optimal strategies under partial information as well as complete information.

The questions on optimal investment strategies for the non-constant volatility $\sigma(\cdot)$ case remain unanswered. This may be an interesting and exciting case since actions of the investor create a trade-off between the increase in the controlled part and the decrease in the precision of the estimates of the unobserved part of the drift. We leave this challenging issue for future study.

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