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Closed-form solutions for options with random initiation under asset price monitoring

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ABSTRACT

This paper studies derivatives to prepare for financial risk from unexpected events. It is difficult for firms and financial institutions to hedge losses triggered by natural catastrophes such as earthquakes, by using derivative securities with fixed initiation and maturities. In this context, we consider an option that is initiated at random by an unexpected event, and moreover, is connected with a barrier of knock-in or knock-out type for asset price monitoring until the time of event. We derive closed-form valuation formulas for these options.

1. Introduction

Financial institutions have been increasingly making use of barrier options, digital options and other path-dependent options. These options are used for a wide range of hedging, risk management and investment purposes as explained in Goldman et al. (1979) and Hull (2006). Furthermore, in recent years many insurance companies have developed more innovative saving products, and these products usually have features of barrier option. In other words, they allow the policyholder to participate in favourable investment performance while maintaining a floor guarantee on the benefit level.

Barrier options have become increasingly popular because they are more flexible and cheaper than vanilla options. Also, they have been created to provide the insurance value of an option without charging as much premium. Many papers provided pricing formulas for various types of barrier options since Merton (1973) (See for example Rich (1997) and Pelsser (2000)). For more complicated barrier options, partial barrier options which is monitored for a part of the option's lifetime are studied in Heynen and Kat (1994) and Hui (1997), where the ending time of monitoring is different from the expiry date of the option. But, the expiry date as well as the ending time of monitoring are predetermined dates.

This paper investigates derivatives which can play a role as an insurance for unexpected events such as climate extremes or earthquakes. Normally, firms and financial institutions hedge risks for their portfolios using derivatives with fixed maturities. If they wish to hedge the financial risk posed by natural catastrophes, they can not utilize contracts such as digital options with fixed starting time and maturity as in the literature on derivatives, because it is uncertain when a catastrophe

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occurs. Jarrow (2010) provided a closed form solution for valuing Cat bonds to manage the risks of catastrophe event losses. Jaimungal and Wang (2006) derived a pricing formula for European catastrophe equity put options under the assumption that the volatility of the stock price is constant and that the amounts of catastrophe losses have a general distribution. Jiang et al. (2013) presented a novel catastrophe option pricing model that considers counterparty risk. In this paper, we consider financial contracts that initiate immediately at a time when a catastrophe occurs, and that have the feature of asset price monitoring with a knock-in or knock-out barrier.

The exponential distribution has been widely used in various categories including life testing, reliability, and operations research owing to its advantages which are nice mathematical form and memoryless property. It is well known that an exponential random variable can be applied to the waiting time until the first event and the lifetime of an item that does not age. See for example Epstein and Sobel (1954) and Embrechts and Schmidli (1994). In this context, we study a derivative which initiates at an exponential random time and matures thereafter, and contains knock-in or knock-out barriers for asset monitoring in advance. Jun and Ku (2013) studied digital barrier options with an exponential random time, in which barrier option is given to an investor after a random event. On the contrary, barrier option is used in this paper for a monitoring purpose before the event.

The outline of the paper is as follows. Section 2 provides the economic rationale for the options considered in this paper. Section 3 presents pricing formulas for digital options linked with down-and-in and down-and-out barrier options, and a graph in Section 3 shows the properties of the solutions. Conclusion is provided in Section 4.

2. Economic rationale

Risk arises from natural events, such as earthquakes, floods and hurricanes. For example, in March 2011, the earthquake in northeast Japan caused a huge property loss and many firms have suffered from the natural disaster. Also, strong earthquakes causing severe damage to the California area have occurred a number of times in the past.

Suppose that a natural catastrophe such as a large earthquake may occur in a developed country, and that a firm in the country is concerned with the financial risk that can be posed by a large earthquake. Then the firm may want to make the following contract with a financial institution as part of their risk management strategy: A digital option will be provided in the event of a large earthquake, i.e., a fixed amount of money will be paid out at maturity (at a fixed time after the event), with the requirement that the underlying asset price never falls to reach a specified barrier until the time of earthquake. This condition may be an indication that the firm has not been in financial difficulties before the event. As long as the barrier is not hit, the contract is kept intact. Also, to reduce the magnitude of the premium the firm might add the condition that the underlying asset price must be under the specified level at maturity. The motivation is that the earthquake actually dealt a severe blow to the firm. Thus, the contract is very useful for the directly and substantially affected firm from the earthquake.

The firm pays a premium to obtain this digital option contract linked with knock-out barrier in preparing emergency. Since this type of contract has the features of both barrier and digital options, it costs much less, but is useful for preventing risks posed by random events like a large earthquake. The pricing formula for such an option is derived in Corollary 3.4.

3. Options with random initiation under asset price monitoring

Suppose that r is the risk-free interest rate and $\sigma > 0$ is constant. We assume the price of the asset S_t follows a geometric Brownian motion $S_t = S_0 \exp\left((r - \sigma^2/2)t + \sigma W_t\right)$ where W_t is a standard Brownian motion under the risk-neutral probability P.

Let $X_t = \frac{1}{\sigma}\ln(S_t/S_0)$ and $\mu = \frac{r}{\sigma} - \frac{\sigma}{2}$. Then $X_t = \mu t + W_t$. We define the minimum and the maximum for X_t to be

$$m_a^b = \inf_{t\in[a,b]} (X_t) \quad \text{and} \quad M_a^b = \sup_{t\in[a,b]} (X_t)$$

and denote by E^P the expectation operator under the P-measure.

Suppose that τ is an exponential random variable with parameter λ and a barrier option with the lifetime of length τ is initiated at time 0, i.e., the monitoring period for asset price barrier is $[0, \tau]$. We define $d = \frac{1}{\sigma}\ln(D/S_0)$, $u = \frac{1}{\sigma}\ln(U/S_0)$ and $k = \frac{1}{\sigma}\ln(K/S_0)$ where $D(\leq S_0)$ is a down barrier, $U(\geq S_0)$ is a up barrier and K is a strike price.

Consider a barrier option of knock-in type followed by another option which pays out the amount of A if the underlying asset price falls to reach the barrier D in $[0, \tau]$ and is greater than the strike price K at time $\tau + T$. The payoff of the option is zero if the underlying asset price does not hit the barrier D or falls below the strike K at time $\tau + T$. We derive a closed-form formula for the price of this option.

Theorem 3.1. *The value V_1 of a digital option connected with a down-and-in option which terminates at exponential random time τ with parameter λ is*

$$V_1 = A\left[a_1 e^{-rT+\left(\mu+\sqrt{2(\lambda+r)+\mu^2}\right)d} N(d_1) - a_1 e^{\lambda T+\left(\mu+\sqrt{2(\lambda+r)+\mu^2}\right)k} N(d_2)\right]$$
$$- A\left(\frac{D}{S_0}\right)^{\frac{2}{\sigma}\sqrt{2(\lambda+r)+\mu^2}} \left[a_2 e^{-rT+\left(\mu-\sqrt{2(\lambda+r)+\mu^2}\right)d} N(d_1) + a_2 e^{\lambda T+\left(\mu-\sqrt{2(\lambda+r)+\mu^2}\right)k} N\left(d_2 + \frac{2k-2d}{\sqrt{T}}\right)\right]$$

where

$$a_1 = \frac{\lambda}{\sqrt{2(\lambda+r)+\mu^2}\left(\mu+\sqrt{2(\lambda+r)+\mu^2}\right)}, \quad a_2 = \frac{\lambda}{\sqrt{2(\lambda+r)+\mu^2}\left(\mu-\sqrt{2(\lambda+r)+\mu^2}\right)},$$

$$d_1 = \frac{d-k+\mu T}{\sqrt{T}}, \qquad d_2 = \frac{d-k-T\sqrt{2(\lambda+r)+\mu^2}}{\sqrt{T}}$$

and $N(\cdot)$ *is the cumulative standard normal distribution function.*

Proof. The value V_1 of this option contract is expressed as

$$\begin{aligned} V_1 &= E^P\left[e^{-r(\tau+T)}A\mathbf{1}_{\{m_0^\tau \le d,\, S_{\tau+T}>K\}}\right] \\ &= A\, E^P\left[e^{-r(\tau+T)}\mathbf{1}_{\{m_0^\tau \le d,\, X_\tau > d,\, X_{\tau+T}>k\}}\right] + A\, E^P\left[e^{-r(\tau+T)}\mathbf{1}_{\{X_\tau \le d,\, X_{\tau+T}>k\}}\right]. \end{aligned} \tag{3.1}$$

Using the joint density of X_t and its running minimum m_0^t at time t (Borodin and Salminen (2002)), the first expectation term in (3.1) is

$$\begin{aligned} &E^P\left[e^{-r(\tau+T)}\mathbf{1}_{\{m_0^\tau \le d,\, X_\tau > d,\, X_{\tau+T}>k\}}\right] \\ &= \int_d^\infty \int_0^\infty \lambda e^{-\lambda t} e^{-r(t+T)} P(m_0^t \le d,\, X_t \in dz_1) P(X_{t+T} > k | z_1) dt \\ &= \int_d^\infty \left(\int_0^\infty \lambda e^{-\lambda t} e^{-r(t+T)} \frac{1}{\sqrt{2\pi t}} e^{\mu z_1 - \frac{\mu^2}{2}t - \frac{(z_1-2d)^2}{2t}} dt \right) P(X_T > k - z_1) dz_1. \end{aligned} \tag{3.2}$$

Laplace transform in Fusai and Roncoroni (2008) is applied to the inner integral in (3.2). Then

$$\begin{aligned} &\int_0^\infty \lambda e^{-\lambda t} e^{-r(t+T)} \frac{1}{\sqrt{2\pi t}} e^{\mu z_1 - \frac{\mu^2}{2}t - \frac{(z_1-2d)^2}{2t}} dt \\ &= \frac{\lambda}{\sqrt{2}} e^{-rT+\mu z_1} \int_0^\infty e^{-(\lambda+r+\frac{\mu^2}{2})t} \frac{1}{\sqrt{\pi t}} e^{-\frac{(\sqrt{2}(z_1-2d))^2}{4t}} dt \\ &= \frac{\lambda e^{-rT}}{\sqrt{2(\lambda+r)+\mu^2}} e^{\mu z_1 - |z_1-2d|\sqrt{2(\lambda+r)+\mu^2}}. \end{aligned}$$

Thus,

$$\begin{aligned} &E^P\left[e^{-r(\tau+T)}\mathbf{1}_{\{m_0^\tau \le d,\, X_\tau > d,\, X_{\tau+T}>k\}}\right] \\ &= e^{2d\sqrt{2(\lambda+r)+\mu^2}} \int_d^\infty \frac{\lambda e^{-rT}}{\sqrt{2(\lambda+r)+\mu^2}} e^{(\mu-\sqrt{2(\lambda+r)+\mu^2})z_1} N\left(\frac{-k+z_1+\mu T}{\sqrt{T}}\right) dz_1 \end{aligned} \tag{3.3}$$

where $N(\cdot)$ is the cumulative standard normal distribution function. In the similar way, the second expectation in (3.1) is

$$\begin{aligned} &E^P\left[e^{-r(\tau+T)}\mathbf{1}_{\{X_\tau \le d,\, X_{\tau+T}>k\}}\right] \\ &= \int_{-\infty}^d \int_0^\infty \lambda e^{-\lambda t} e^{-r(t+T)} P(X_t \in dz_2) P(X_{t+T} > k | z_2) dt \\ &= \int_{-\infty}^d \frac{\lambda}{\sqrt{2}} e^{-rT+\mu z_2} \int_0^\infty e^{-(\lambda+r+\frac{\mu^2}{2})t} \frac{1}{\sqrt{\pi t}} e^{-\frac{(\sqrt{2}z_2)^2}{4t}} dt\, P(X_T > k - z_2) dz_2 \\ &= \int_{-\infty}^d \frac{\lambda e^{-rT}}{\sqrt{2(\lambda+r)+\mu^2}} e^{(\mu+\sqrt{2(\lambda+r)+\mu^2})z_2} N\left(\frac{-k+z_2+\mu T}{\sqrt{T}}\right) dz_2. \end{aligned} \tag{3.4}$$

Let

$$a_1 = \frac{\lambda}{\sqrt{2(\lambda+r)+\mu^2}\left(\mu+\sqrt{2(\lambda+r)+\mu^2}\right)}$$

and

$$a_2 = \frac{\lambda}{\sqrt{2(\lambda+r)+\mu^2}\left(\mu-\sqrt{2(\lambda+r)+\mu^2}\right)}.$$

Apply the integration by parts to the integrals in (3.3) and (3.4). Then, we obtain that (3.3) is equal to

$$e^{2d\sqrt{2(\lambda+r)+\mu^2}-rT}\Bigg[-a_2 e^{\left(\mu-\sqrt{2(\lambda+r)+\mu^2}\right)d} N\left(\frac{d-k+\mu T}{\sqrt{T}}\right)$$
$$-\int_d^\infty a_2 \frac{1}{\sqrt{2\pi T}} \exp\left\{\left(\mu-\sqrt{2(\lambda+r)+\mu^2}\right)z_1 - \frac{1}{2}\left(\frac{-k+z_1+\mu T}{\sqrt{T}}\right)^2\right\}dz_1\Bigg]$$
$$= e^{2d\sqrt{2(\lambda+r)+\mu^2}-rT}\Bigg[-a_2 e^{\left(\mu-\sqrt{2(\lambda+r)+\mu^2}\right)d} N\left(\frac{d-k+\mu T}{\sqrt{T}}\right)$$
$$-a_2 e^{(\lambda+r)T+\left(\mu-\sqrt{2(\lambda+r)+\mu^2}\right)k}\int_d^\infty \frac{1}{\sqrt{2\pi T}} \exp\left\{-\frac{1}{2}\left(\frac{z_1-k+T\sqrt{2(\lambda+r)+\mu^2}}{\sqrt{T}}\right)^2\right\}dz_1\Bigg]$$
$$= -e^{2d\sqrt{2(\lambda+r)+\mu^2}-rT}\Bigg[a_2 e^{\left(\mu-\sqrt{2(\lambda+r)+\mu^2}\right)d} N\left(\frac{d-k+\mu T}{\sqrt{T}}\right)$$
$$+a_2 e^{(\lambda+r)T+\left(\mu-\sqrt{2(\lambda+r)+\mu^2}\right)k} N\left(\frac{-d+k-T\sqrt{2(\lambda+r)+\mu^2}}{\sqrt{T}}\right)\Bigg]$$

and also (3.4) becomes

$$e^{-rT} a_1 e^{\left(\mu+\sqrt{2(\lambda+r)+\mu^2}\right)d} N\left(\frac{d-k+\mu T}{\sqrt{T}}\right)$$
$$- e^{-rT} a_1 e^{(\lambda+r)T+\left(\mu+\sqrt{2(\lambda+r)+\mu^2}\right)k} N\left(\frac{d-k-T\sqrt{2(\lambda+r)+\mu^2}}{\sqrt{T}}\right).$$

Collecting the terms, the proof is completed.

□

As a special case, one can obtain the price for an option where the amount of A is paid if the asset price exceeds the strike K at time $\tau+T$, and the payoff is zero otherwise. By putting $D=S_0$ in Theorem 3.1, we obtain the following result.

Corollary 3.2. *The value V_2 of a digital option which starts at exponential random time τ with parameter λ is*

$$V_2 = \frac{\lambda A}{\lambda+r} e^{-rT} N\left(d_1 - \frac{d}{\sqrt{T}}\right)$$
$$- A e^{\lambda T}\left[a_1 e^{\left(\mu+\sqrt{2(\lambda+r)+\mu^2}\right)k} N\left(d_2 - \frac{d}{\sqrt{T}}\right) + a_2 e^{\left(\mu-\sqrt{2(\lambda+r)+\mu^2}\right)k} N\left(d_2 + \frac{2k-d}{\sqrt{T}}\right)\right].$$

We next consider a barrier option of knock-out type followed by an additional option which pays out the amount of A if the underlying asset price does not cross the barrier D in $[0,\tau]$ and is greater than the strike price K at time $\tau+T$. This option pays off zero when the underlying asset price hits the barrier D or falls below the strike K at $\tau+T$.

Theorem 3.3. *The value V_3 of a digital option connected with a down-and-out option which terminates at exponential random time τ with parameter λ is*

$$V_3 = A\Bigg[\frac{\lambda}{\lambda+r} e^{-rT} N\left(d_1 - \frac{d}{\sqrt{T}}\right) - a_1 e^{-rT+\left(\mu+\sqrt{2(\lambda+r)+\mu^2}\right)d} N(d_1)$$
$$-a_1 e^{\lambda T+\left(\mu+\sqrt{2(\lambda+r)+\mu^2}\right)k}\left\{N\left(d_2 - \frac{d}{\sqrt{T}}\right) - N(d_2)\right\} - a_2 e^{\lambda T+\left(\mu-\sqrt{2(\lambda+r)+\mu^2}\right)k} N\left(d_2 + \frac{2k-d}{\sqrt{T}}\right)\Bigg]$$
$$+ A\left(\frac{D}{S_0}\right)^{\frac{2}{\sigma}\sqrt{2(\lambda+r)+\mu^2}}\left[a_2 e^{-rT+\left(\mu-\sqrt{2(\lambda+r)+\mu^2}\right)d} N(d_1) + a_2 e^{\lambda T+\left(\mu-\sqrt{2(\lambda+r)+\mu^2}\right)k} N\left(d_2 + \frac{2k-2d}{\sqrt{T}}\right)\right]$$

where $N(\,\cdot\,)$ is the cumulative standard normal distribution function.

Proof. The value V_3 of this contract is expressed as

$$V_3 = E^P\left[e^{-r(\tau+T)} A \mathbf{1}_{\{m_0^\tau > d,\, S_{\tau+T} > K\}}\right]$$
$$= A\, E^P\left[e^{-r(\tau+T)} \mathbf{1}_{\{X_\tau > d,\, X_{\tau+T} > k\}}\right] - A\, E^P\left[e^{-r(\tau+T)} \mathbf{1}_{\{m_0^\tau \le d,\, X_\tau > d,\, X_{\tau+T} > k\}}\right]$$

We use similar techniques used in the proof of Theorem 3.1. Also note that $a_1 - a_2 = \frac{\lambda}{\lambda+r}$.

$$\begin{aligned}
&E^P\left[e^{-r(\tau+T)}\mathbf{1}_{\{X_\tau>d,\, X_{\tau+T}>k\}}\right] \\
&= \int_d^0 \frac{\lambda e^{-rT}}{\sqrt{2(\lambda+r)+\mu^2}} e^{(\mu+\sqrt{2(\lambda+r)+\mu^2})z} N\left(\frac{-k+z+\mu T}{\sqrt{T}}\right)dz \\
&\quad + \int_0^\infty \frac{\lambda e^{-rT}}{\sqrt{2(\lambda+r)+\mu^2}} e^{(\mu-\sqrt{2(\lambda+r)+\mu^2})z} N\left(\frac{-k+z+\mu T}{\sqrt{T}}\right)dz \\
&= \frac{\lambda}{\lambda+r}e^{-rT}N\left(\frac{-k+\mu T}{\sqrt{T}}\right) - e^{-rT}a_1 e^{\left(\mu+\sqrt{2(\lambda+r)+\mu^2}\right)d} N\left(\frac{d-k+\mu T}{\sqrt{T}}\right) \\
&\quad - a_1 e^{\lambda T+\left(\mu+\sqrt{2(\lambda+r)+\mu^2}\right)k} N\left(\frac{-k-T\sqrt{2(\lambda+r)+\mu^2}}{\sqrt{T}}\right) \\
&\quad + a_1 e^{\lambda T+\left(\mu+\sqrt{2(\lambda+r)+\mu^2}\right)k} N\left(\frac{d-k-T\sqrt{2(\lambda+r)+\mu^2}}{\sqrt{T}}\right) \\
&\quad - a_2 e^{\lambda T+\left(\mu-\sqrt{2(\lambda+r)+\mu^2}\right)k} N\left(\frac{k-T\sqrt{2(\lambda+r)+\mu^2}}{\sqrt{T}}\right).
\end{aligned}$$

By combining with the terms for $E^P\left[e^{-r(\tau+T)}\mathbf{1}_{\{m_0^\tau \le d,\, X_\tau>d,\, X_{\tau+T}>k\}}\right]$ provided in the proof of Theorem 3.1, the desired result is achieved.

□

Consider a financial contract paying the amount of A when the underlying asset price never falls to reach the barrier D in $[0, \tau]$ and is lower than the strike price K at time . The following formula is now obtained by using V_3 in Theorem 3.3.

Corollary 3.4. *The value V_4 of a digital option connected with a down-and-out option which terminates at exponential random time τ with parameter λ is given by*

$$V_4 = \frac{\lambda A}{\lambda+r}e^{-rT}\left[1 - e^{\left(\mu+\sqrt{2(\lambda+r)+\mu^2}\right)d}\right] - V_3.$$

Proof. The value V_4 is

$$\begin{aligned}
V_4 &= E^P\left[e^{-r(\tau+T)}A\mathbf{1}_{\{m_0^\tau>d,\, S_{\tau+T}<K\}}\right] \\
&= A\,E^P\left[e^{-r(\tau+T)}\mathbf{1}_{\{m_0^\tau>d,\, X_{\tau+T}<k\}}\right] \\
&= A\,E^P\left[e^{-r(\tau+T)}\mathbf{1}_{\{m_0^\tau>d\}}\right] - A\,E^P\left[e^{-r(\tau+T)}\mathbf{1}_{\{m_0^\tau>d,\, X_{\tau+T}\ge k\}}\right] \\
&= A\,E^P\left[e^{-r(\tau+T)}\left(1-\mathbf{1}_{\{m_0^\tau\le d\}}\right)\right] - V_3 \\
&= \frac{\lambda A}{\lambda+r}e^{-rT}\left[1 - e^{\left(\mu+\sqrt{2(\lambda+r)+\mu^2}\right)d}\right] - V_3.
\end{aligned}$$

□

Fig. 1 shows how the value V_4 in Corollary 3.4 changes when λ varies from 0 to 2 and D varies from 80 to 100. The parameter values that we used are $S_0 = 100$, $K = 80$, $A = 10^6$, $\sigma = 0.3$, $r = 0.05$ and $T = 0.5$.

If the exponential parameter λ is large, τ is more likely to be small. Then, the probability of the underlying asset falling to reach the barrier D in the period $[0, \tau]$ becomes small. Thus the option value V_4 with a barrier of knock-out type increases as λ grows. Also, if the down barrier D takes the larger value, the option price V_4 with knock-out barrier is smaller, as observed in Fig. 1. This property is the same as that of the regular barrier option.

Table 1 shows a comparison between the simulation results and the exact values from our pricing formula for different values of λ and down barrier D. MC represents the results from Monte Carlo simulation using Antithetic Variates, the Variance Reduction Method of Monte Carlo simulation. The parameter values in this computation are $S_0 = 100$, $K = 100$, $A = 1$, $T = 1$, $\sigma = 0.3$ and $r = 0.05$. Monte Carlo method requires much larger amount of computer time because a large number of sample paths and exponential random times, and a large enough monitoring frequency must be needed in order to catch the hitting times. For the Monte Carlo simulation results in Table 1, a monitoring frequency is 1000, the number of sample paths is 10000, and the number of exponential random times is 10000. We note that the computation time of each value for Monte Carlo simulation was a couple hours while our closed-form formula can be computed in less than a second.

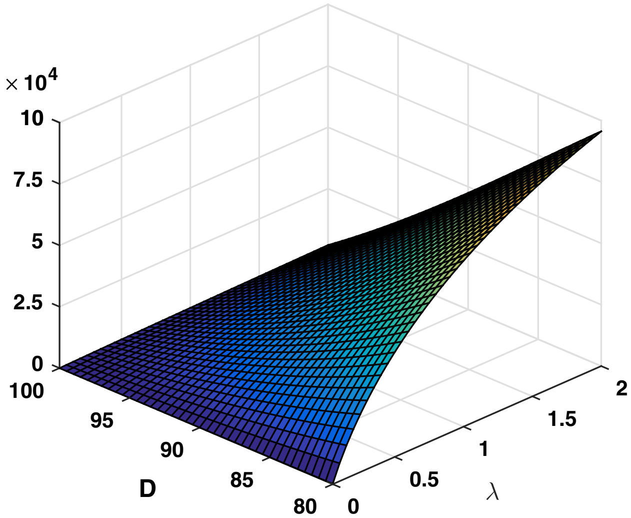

Fig. 1. The value V_4 connected with a knock-out option when λ and down barrier D vary.

Table 1
Comparison of formula and Monte Carlo Simulation for V_1 and V_4

λ	D	V_1		V_4	
		Formula	MC	Formula	MC
0.5	70	0.0550	0.0525	0.2345	0.2344
	80	0.1253	0.1218	0.1570	0.1565
	90	0.2484	0.2424	0.0767	0.0763
1	70	0.0304	0.0291	0.3166	0.3150
	80	0.0894	0.0871	0.2297	0.2295
	90	0.2182	0.2135	0.1216	0.1213
2	70	0.0135	0.0138	0.3881	0.3902
	80	0.0553	0.0546	0.3086	0.3068
	90	0.1772	0.1739	0.1807	0.1806

Note that the economic explanation for V_4 is provided in Section 2. The economic (financial) explanations for V_1, V_2 and V_3 can be given as follows: For example, consider that a firm wants to buy a derivative to hedge risks for the rising price of oil, which are triggered by natural catastrophes in a large oil-producing country. If the oil price is likely to decrease in the short term, a barrier of knock-in type until the time of event can prevent the firm from paying extra premium. The pricing formula V_1 for such an option is derived in Theorem 3.1. As a special case, V_2 gives the pricing formula when the firm wants this digital option contract without the feature of barrier monitoring. In a similar situation, V_3 provides the pricing formula for this digital option contract linked with a knock-out barrier.

If one might want to require, in addition to the conditions for V_4, that the underlying asset price continuously decreases and doesn't exceed a barrier line for the period of $[\tau, \tau + T]$, we provide the following valuation formula.

Theorem 3.5. *The value V_5 of a digital option having a up-and-out barrier connected with a down-and-out option which terminates at exponential random time τ with parameter λ is*

$$\begin{aligned} V_5 =& V_4 - A\left(\frac{U^2}{DS_0}\right)^{\frac{\mu}{\sigma}}(a_1 - a_2)e^{-rT + d\sqrt{2(\lambda+r)+\mu^2}}N(d_3) \\ &+ Ae^{\lambda T}\left(\frac{K}{S_0}\right)^{\frac{\mu}{\sigma}}\left[a_1 e^{(-2u+2d+k)\sqrt{2(\lambda+r)+\mu^2}} - a_2 e^{(2u-k)\sqrt{2(\lambda+r)+\mu^2}}\right]N(d_4) \end{aligned}$$

where

$$d_3 = \frac{-2u + d + k - \mu T}{\sqrt{T}} \quad \text{and} \quad d_4 = \frac{-d + 2u - k - T\sqrt{2(\lambda + r) + \mu^2}}{\sqrt{T}}.$$

Proof. The value V_5 can be calculated by essentially the same techniques as in the proofs of Theorem 3.1 and 3.3.

$$\begin{aligned}
V_5 &= E^P\left[e^{-r(\tau+T)}A\mathbf{1}_{\{m_0^\tau>d,\, S_{\tau+T}<K,\, M_\tau^{\tau+T}<u\}}\right]\\
&= A\,E^P\left[e^{-r(\tau+T)}\mathbf{1}_{\{m_0^\tau>d,\, X_{\tau+T}<k\}}\right] - A\,E^P\left[e^{-r(\tau+T)}\mathbf{1}_{\{m_0^\tau>d,\, X_{\tau+T}<k,\, M_\tau^{\tau+T}\geq u\}}\right]\\
&= V_4 - A\,E^P\left[e^{-r(\tau+T)}\mathbf{1}_{\{X_\tau>d,\, X_{\tau+T}<k,\, M_\tau^{\tau+T}\geq u\}}\right] + A\,E^P\left[e^{-r(\tau+T)}\mathbf{1}_{\{m_0^\tau\leq d,\, X_\tau>d,\, X_{\tau+T}<k,\, M_\tau^{\tau+T}\geq u\}}\right]\\
&= V_4 - A\left(\frac{U^2}{DS_0}\right)^{\frac{\mu}{\sigma}}(a_1-a_2)e^{-rT+d\sqrt{2(\lambda+r)+\mu^2}}N\left(\frac{-2u+d+k-\mu T}{\sqrt{T}}\right)\\
&\quad + Ae^{\lambda T}\left(\frac{K}{S_0}\right)^{\frac{\mu}{\sigma}}\left[a_1e^{(-2u+2d+k)\sqrt{2(\lambda+r)+\mu^2}} - a_2e^{(2u-k)\sqrt{2(\lambda+r)+\mu^2}}\right]\times N\left(\frac{-d+2u-k-T\sqrt{2(\lambda+r)+\mu^2}}{\sqrt{T}}\right).
\end{aligned}$$

□

Remark 3.6. The pricing formulas derived in this section remain the same at any time $t < \tau$ due to the memoryless property of exponential distribution.

4. Conclusions

This paper considers derivatives to hedge financial risk that may arise randomly. It is difficult to hedge exposure to catastrophe loss by using derivative securities with fixed time horizon. With this motivation, derivatives that terminate or initiate at random times are needed in practice. In this paper, we derive closed-form valuation formulas for options which initiate at a random time with exponential distribution and contain knock-in or knock-out barriers for asset monitoring in advance. Due to this contribution, financial institutions have benefits from this research to utilize and price such products easily.

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