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Static hedging of chained-type barrier options



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ABSTRACT

This paper concerns barrier options which are chained together. When the underlying asset price hits a certain barrier level, another barrier option is given to a primary option holder. Then, if the asset price hits another barrier, a third barrier option is given, and so on. The paper studies the hedging problem for these chained-type barrier options. We use the (double) reflection principle and propose a static replication portfolio of vanilla options for hedging of these options in the Black–Scholes model. The Monte Carlo simulation results for vanilla options with adjusted payoffs are provided to demonstrate the accuracy of the hedging strategies. A comparison between static hedging and delta hedging for a chained barrier option shows static hedge performs better than delta hedge.

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1. Introduction

A lot of exotic options have been traded in the financial market. In particular, barrier options are one of the most heavily traded exotic derivatives in the over-the-counter market, since they have lower prices than the plain vanilla options.

Many papers provided pricing formulas for various types of single barrier options; for example Merton (1973), Reiner and Rubinstein (1991), and Rich (1997). For more complicated barrier options, Geman and Yor (1996), Kunitomo and Ikeda (1992), and Pelsser (2000) researched on the double barrier option price. Heynen and Kat (1994) studied partial barrier options where the underlying price is

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monitored for a part of the option's lifetime. As a natural variation on the partial barrier structure, window barrier options have become popular with investors, particularly in foreign exchange markets. For a window barrier option, a monitoring period for the barrier commences at the forward starting date and terminates at the early ending date. (We refer to [Hui \(1997\)](#) and [Guillaume \(2003\)](#).) Moreover, non-Black–Scholes models have been considered in pricing options ([Lian, Liao, & Chen, 2015](#); [Lin, Huang, & Li, 2015](#)). In particular, non-Black–Scholes models have been adopted to study barrier options; for instance, the constant elasticity of variance (CEV) model in [Boyle and Tian \(1999\)](#) and a jump diffusion model in [Kou and Wang \(2004\)](#).

From a bank's point of view, an important issue is how to hedge these barrier options. Options traders ordinarily hedge options by shorting the dynamic hedging portfolio against a long position in the option to eliminate all the risk related to stock price movement. The delta hedging method is also known as dynamic hedging which continuously adjusts the weights in a portfolio consisting of risky stock and risk-free zero-coupon bonds as time passes or the stock price moves. The performance of various hedge strategies using a variety of models has been studied by a number of papers including [Bakshi, Cao, and Chen \(1997\)](#), [Dumas, Fleming, and Whaley \(1998\)](#), [Garcia and Gencay \(2000\)](#) for equity underlyings. These studies focused on hedging vanilla options.

There are some problems on hedging barrier options with the delta hedging method. Delta hedging only works well under the condition of small changes in the stock prices. However, when the time is close to maturity T or underlying asset price is in the barrier region, it is very difficult to delta hedge the options. Also, continuous weight adjustment is impossible in practice and there are transaction costs associated with adjusting the portfolio weights which would grow with frequency of adjustment.

In the case of barrier options, there is an alternative way to hedge, which is the static hedging method. In the static hedging portfolio, given a particular target option, we can construct a portfolio of standard options with varying strikes, maturities and fixed weights, which will not require any further adjustment (unless the options knock out (or in)) and will exactly replicate the value of the target option. [Tompkins \(2002\)](#) showed that static hedging is less sensitive to the model risk such as volatility misspecification. Although the bid-ask spreads involved in option trading are typically an order of magnitude larger than for trading in the underlying, [Siven and Poulsen \(2008\)](#) demonstrated that with realistic frictions, static hedging with options is still the best way to reduce the risk of large losses. However, static hedging does not alleviate the exploding Greeks problem of discontinuous barrier options, such as a down-and-out call. [Nalholm and Poulsen \(2006\)](#) demonstrated that careful choice of hedge instruments as well as regularization techniques is necessary to achieve practically feasible results.

There are two kinds of static hedges of barrier options, the calendar-spread and the strike-spread approaches. The calendar-spread approach is to construct a hedging portfolio composed of standard options with different maturities. [Derman, Ergener, and Kani \(1995\)](#) showed how to create static hedge portfolios for barrier options using plain vanilla options with different expiry dates. [Chou and Georgiev \(1998\)](#) showed how a static replication portfolio of options with a single maturity and multiple strikes can be converted into a static replication portfolio of options with a single strike and multiple maturities under the [Black and Scholes \(1973\)](#) (BS) model. [Chung, Shin, and Tsai \(2010\)](#) modified the static replication approach of [Derman et al. \(1995\)](#) to hedge continuous barrier options under the BS model. The strike-spread one is to construct a hedging portfolio with standard options with different strikes. This approach was introduced by [Bowie and Carr \(1994\)](#), [Carr and Chou \(1997\)](#), and [Carr, Ellis, and Gupta \(1998\)](#) assuming the BS model. A good overview of this approach is given in [Poulsen \(2006\)](#). The point is to convert the problem of replicating a barrier option to a problem of replicating a European security, which turns out to have a non-linear payoff function.

All papers mentioned above are concerned with static hedging of barrier options where monitoring of the barrier starts at a predetermined date. However, this paper concerns static hedging of barrier options where monitoring of the other barrier starts at a random time when the underlying asset price first crosses a certain barrier level. Such options have been studied in [Jun and Ku \(2012, 2013\)](#) where the authors derived the closed form valuation formulae for chained options with constant and exponential barriers. These options have become popular in the Japanese OTC equity and FX markets. In this paper, we obey the strike-spread approach under the BS model and replicate these chained options.

The outline of the paper is as follows. In Section 2, we derive a necessary lemma for the static hedging of barrier options starting at hitting time and introduce the replication strategies for a down-and-in option (DI_u) and a down-and-out option (DO_u) activated at a time when the underlying asset price hits an upper barrier level. Section 3 gives a more detailed analysis of adjusted payoffs in the case of call option. We show the graphs of adjusted payoffs at time 0 and at the time when the upper barrier is hit. In Section 4, we provide the Monte Carlo simulation results for the values of vanilla options with adjusted payoffs, then compare them with the exact option values. In addition, a comparison between static hedge and delta hedge for a down-and-in call option is presented. Section 5 gives some conclusions.

2. Static replication of chained options

Let r be the risk-free interest rate, d be a dividend rate, and $\sigma > 0$ be a constant. We assume the price of the asset S_t follows a geometric Brownian motion

$$S_t = S_0 \exp((r - d - \sigma^2/2)t + \sigma W_t)$$

where W_t is a standard Brownian motion. We fix a down-barrier $D(< S_0)$ and an up-barrier $U(> S_0)$.

The following lemma plays an important role in our static hedging analysis.

Lemma 2.1. Suppose that X is a portfolio of European options expiring at time T with payoff:

$$X(S_T) = \begin{cases} f(S_T) & \text{if } S_T \in (A, B) \\ 0 & \text{otherwise} \end{cases}$$

For $H > 0$, let Y be a portfolio of European options with maturity T and payoff:

$$Y(S_T) = \begin{cases} \left(\frac{S_T}{H}\right)^p f\left(\frac{H^2}{S_T}\right) & \text{if } S_T \in \left(\frac{H^2}{B}, \frac{H^2}{A}\right) \\ 0 & \text{otherwise} \end{cases}$$

For $G > 0$, let Z be a portfolio of European options with maturity T and payoff:

$$Z(S_T) = \begin{cases} \left(\frac{G}{H}\right)^p f\left(\frac{H^2 S_T}{G^2}\right) & \text{if } S_T \in \left(\frac{AG^2}{H^2}, \frac{BG^2}{H^2}\right) \\ 0 & \text{otherwise} \end{cases}$$

where the power $p \equiv 1 - 2(r - d)/\sigma^2$ and r, d, σ are the interest rate, dividend rate and volatility rate, respectively.

Then, X and Y have the same value whenever the spot equals H . Also, Y and Z have the same value whenever the spot equals G .

Proof. By the risk-neutral valuation, the value of X when the spot is H at time t_2 is expressed as

$$V_X(H, t_2) = e^{-rt_2} \int_A^B f(S_T) \frac{1}{S_T \sqrt{2\pi\sigma^2\tau_2}} \exp\left[-\frac{(\ln(S_T/H) - (r - d - (1/2)\sigma^2)\tau_2)^2}{2\sigma^2\tau_2}\right] dS_T$$

where $\tau_2 = T - t_2$. Let $\tilde{S}_T = H^2/S_T$. Then, $dS_T = -H^2/\tilde{S}_T^2 d\tilde{S}_T$ and

$$V_X(H, t_2) = e^{-rt_2} \int_{H^2/B}^{H^2/A} \left(\frac{\tilde{S}_T}{H}\right)^p f\left(\frac{H^2}{\tilde{S}_T}\right) \frac{1}{\tilde{S}_T \sqrt{2\pi\sigma^2\tau_2}} \exp\left[-\frac{(\ln(\tilde{S}_T/H) - (r - d - (1/2)\sigma^2)\tau_2)^2}{2\sigma^2\tau_2}\right] d\tilde{S}_T$$

where $p \equiv 1 - 2(r - d)/\sigma^2$. $V_X(H, t_2)$ is equal to $V_Y(H, t_2)$ which is the value of Y when the spot is H at time t_2 . Also, let $\hat{S}_T = G^2/\tilde{S}_T$. Then $d\tilde{S}_T = -G^2/\hat{S}_T^2 d\hat{S}_T$ and the value of Y when the spot is G at time t_1 ,

$$\begin{aligned}
V_Y(G, t_1) &= e^{-r\tau_1} \int_{\frac{H^2}{\tilde{S}_T}}^{\frac{H^2}{\tilde{S}_T}} \left(\frac{\tilde{S}_T}{H} \right)^p f \left(\frac{H^2}{\tilde{S}_T} \right) \frac{1}{\tilde{S}_T \sqrt{2\pi\sigma^2\tau_1}} \exp \left[-\frac{(\ln(\tilde{S}_T/G) - (r-d - (1/2)\sigma^2)\tau_1)^2}{2\sigma^2\tau_1} \right] d\tilde{S}_T \\
&= e^{-r\tau_1} \int_{\frac{H^2}{\tilde{S}_T}}^{\frac{H^2}{\tilde{S}_T}} \left(\frac{G^2}{H\tilde{S}_T} \right)^p f \left(\frac{H^2\tilde{S}_T}{G^2} \right) \frac{1}{\tilde{S}_T \sqrt{2\pi\sigma^2\tau_1}} \exp \left[-\frac{(\ln(G/\tilde{S}_T) - (r-d - (1/2)\sigma^2)\tau_1)^2}{2\sigma^2\tau_1} \right] d\tilde{S}_T \\
&= e^{-r\tau_1} \int_{\frac{H^2}{\tilde{S}_T}}^{\frac{H^2}{\tilde{S}_T}} \left(\frac{G}{H} \right)^p f \left(\frac{H^2\tilde{S}_T}{G^2} \right) \frac{1}{\tilde{S}_T \sqrt{2\pi\sigma^2\tau_1}} \exp \left[-\frac{(\ln(\tilde{S}_T/G) - (r-d - (1/2)\sigma^2)\tau_1)^2}{2\sigma^2\tau_1} \right] d\tilde{S}_T
\end{aligned}$$

where $\tau_1 = T - t_1$. Thus, $V_Y(G, t_1)$ is equal to $V_Z(G, t_1)$, which is the value of Z when the spot is G at time t_1 . $\square$

Suppose there are two barriers, upper barrier and lower barrier, that straddle the initial spot price. When the two barriers lie in the same direction with respect to the initial spot price, the chained barrier option becomes a general single barrier option with the farther one from the initial spot price among the same direction barriers.

In the following theorem, we derive the static replication for a chained option where a down-and-in option commences at the time when the underlying asset price hits the upper barrier. We note that the payoff of any European security can be replicated with positions in zero-coupon bonds, forwards and vanilla European put and call options (see Carr & Picron, 1999).

Theorem 2.2. Let DI_u be a down-and-in chained option with lower barrier D , maturity T , and payoff $f(S_T)$ at maturity under the condition that upper barrier U was touched first. Then, there exists a two-step single-maturity static replication strategy for DI_u , where the replicas have payoff $\hat{f}(S_T)$ at maturity T :

$$\hat{f}^{DI_u}(S_T) = \begin{cases} \left(\frac{S_T}{U} \right)^p f \left(\frac{U^2}{S_T} \right) + \left(\frac{U}{D} \right)^p f \left(\frac{D^2 S_T}{U^2} \right) & \text{if } S_T > \frac{U^2}{D} \\ 0 & \text{otherwise} \end{cases}$$

where $p = 1 - 2(r-d)/\sigma^2$.

Proof. A down-and-in chained option pays $f(S_T)$ at time T if upper barrier U has been hit over $[0, T]$ and lower barrier D has been hit after the upper barrier had been hit. Define the first hitting time of the upper barrier, $t_1 = \min \{t : S_t = U\}$, and the first hitting time of the lower barrier after the upper barrier is hit, $t_2 = \min \{t > t_1 : S_t = D\}$. Then the down-and-in chained option value at time $t_0 < t_1$ is

$$DI_u(t_0) = e^{-r(T-t_0)} E^Q [f(S_T) \mathbf{1}_{\{t_1 < t_2 < T\}}] \quad (2.1)$$

where $\mathbf{1}_{\{\cdot\}}$ is an indicator function and the expectation is taken under the risk-neutral probability Q .

We call $\hat{f}(S_T)$ the adjusted payoff for the down-and-in chained option under the condition that up-barrier was touched first. Define the adjusted payoff with maturity T as:

$$\hat{f}^{DI_u}(S_T) = \begin{cases} \left(\frac{S_T}{U} \right)^p f \left(\frac{U^2}{S_T} \right) + \left(\frac{U}{D} \right)^p f \left(\frac{D^2 S_T}{U^2} \right) & \text{if } S_T > \frac{U^2}{D} \\ 0 & \text{otherwise} \end{cases} \quad (2.2)$$

where $p \equiv 1 - 2(r-d)/\sigma^2$.

We rebalance at times t_1 and t_2 . If the upper barrier is never hit, then a portfolio of vanilla options with payoff (2.2) at maturity T expires worthless. If the upper barrier is hit, then Lemma 2.1 indicates that at the first hitting time of the upper barrier, the value of $\left(\frac{S_T}{U} \right)^p f \left(\frac{U^2}{S_T} \right) \mathbf{1}_{\{S_T > \frac{U^2}{D}\}}$, the upper first term of (2.2), matches the value of a payoff $f(S_T) \mathbf{1}_{\{S_T < D\}}$ and the value of $\left(\frac{U}{D} \right)^p f \left(\frac{D^2 S_T}{U^2} \right) \mathbf{1}_{\{S_T > \frac{U^2}{D}\}}$, the

upper second term of (2.2), matches the value of a payoff $\left(\frac{S_T}{D}\right)^p f\left(\frac{D^2}{S_T}\right) \mathbf{1}_{\{S_T < D\}}$. Thus, a portfolio of vanilla options with payoff $\widehat{f}^{DI_u}(S_T)$ have the same value as a portfolio with payoff

$$\widehat{f}^{DI_u}(S_T) = \begin{cases} 0 & \text{otherwise} \\ f(S_T) + \left(\frac{S_T}{D}\right)^p f\left(\frac{D^2}{S_T}\right) & \text{if } S_T < D \end{cases} \quad (2.3)$$

After time t_1 , if the lower barrier is not hit, then a portfolio of vanilla options with payoff (2.3) expires worthless. When the lower barrier is hit at time t_2 , from Lemma 2.1, the value of $\left(\frac{S_T}{D}\right)^p f\left(\frac{D^2}{S_T}\right) \mathbf{1}_{\{S_T < D\}}$, the lower second term of (2.3), matches the value of a payoff $f(S_T) \mathbf{1}_{\{S_T > D\}}$.

More concretely, our replicating strategy is:

1. At initiation, purchase a portfolio of European options that provides payoff $\widehat{f}^{DI_u}(S_T)$ at maturity T .
2. If the upper barrier is not hit before time T , the portfolio is liquidated.
3. If the upper barrier is hit before time T , then at time t_1 , sell the portfolio of European options that provides payoff $\widehat{f}^{DI_u}(S_T)$ and buy a portfolio of European options that provides payoff $\widehat{f}^{DI_u}(S_T)$.
4. If the lower barrier is not hit between t_1 and T , the portfolio is liquidated. Otherwise, at time t_2 when the lower barrier is hit, sell the portfolio of European options that provides payoff $\left(\frac{S_T}{D}\right)^p f\left(\frac{D^2}{S_T}\right) \mathbf{1}_{\{S_T < D\}}$ and buy a portfolio of European options that provides payoff $f(S_T) \mathbf{1}_{\{S_T > D\}}$.

Note that when the portfolio is liquidated, it has zero value.

Therefore, after rebalancing at two hitting times t_1 and t_2 , the total portfolio of options delivers a payoff of $f(S_T)$ as required. As a result, we can replicate a down-and-in chained option DI_u with vanilla options. Also, the option value is given by

$$DI_u(t_0) = e^{-r(T-t_0)} E^Q[\widehat{f}^{DI_u}(S_T)] \quad (2.4)$$

where the expectation is taken under the unique risk-neutral measure Q . $\square$

The following theorem presents the static replication for a down-and-out chained option under the condition that up-barrier U was touched first.

Theorem 2.3. Let DO_u be a down-and-out chained option with lower barrier D , maturity T , and payoff $f(S_T)$ at maturity under the condition that upper barrier U was touched first. Then, there exists a two-step single-maturity static replication strategy for DO_u , where the replicas have payoff $\widehat{f}^{DO_u}(S_T)$ at maturity T :

$$\widehat{f}^{DO_u}(S_T) = \left[f(S_T) + \left(\frac{S_T}{U}\right)^p f\left(\frac{U^2}{S_T}\right) \right] \mathbf{1}_{\{S_T > U\}} - \left[\left(\frac{S_T}{U}\right)^p f\left(\frac{U^2}{S_T}\right) + \left(\frac{U}{D}\right)^p f\left(\frac{D^2 S_T}{U^2}\right) \right] \mathbf{1}_{\{S_T > \frac{U^2}{D}\}}$$

where $p = 1 - 2(r - d)/\sigma^2$.

Proof. The down-and-out chained option value at time $t_0 < t_1$ is

$$DO_u(t_0) = e^{-r(T-t_0)} E^Q[f(S_T) \mathbf{1}_{\{t_1 < T < t_2\}}].$$

The sum of the payoffs for a down-and-in chained option and a down-and-out chained option is equal to the payoff of the corresponding standard up-and-in option. Thus

$$f(S_T) \mathbf{1}_{\{t_1 < T < t_2\}} = f(S_T) \mathbf{1}_{\{t_1 < T\}} - f(S_T) \mathbf{1}_{\{t_1 < t_2 < T\}}$$

We obtain the adjusted payoff of DO_u by subtracting (2.2) from the adjusted payoff of the corresponding standard up-and-in option given in Carr and Chou (1997). $\square$

The following corollary is concerned with up-and-in and up-and-out options under the condition that lower barrier was touched first. We obtain the results by exchanging the roles of upper barrier U and lower barrier D in Theorems 2.2 and 2.3.

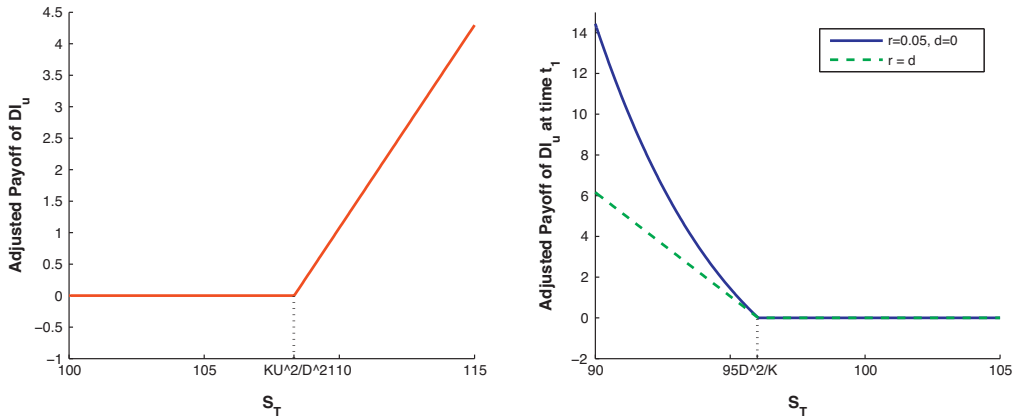


Fig. 1. Adjusted payoff of DI_u with $f(S_T) = (S_T - K)^+$.

Corollary 2.4. Let UI_d be a up-and-in chained option with upper barrier U , maturity T , and payoff $f(S_T)$ at maturity under the condition that lower barrier D was touched first. Then, there exists a two-step single-maturity static replication strategy for UI_d where the replicas have payoff $\hat{f}^{UI_d}(S_T)$ at maturity T :

$$\hat{f}^{UI_d}(S_T) = \left[\left(\frac{S_T}{D} \right)^p f\left(\frac{D^2}{S_T}\right) + \left(\frac{D}{U} \right)^p f\left(\frac{U^2 S_T}{D^2}\right) \right] \mathbf{1}_{\{S_T < \frac{D^2}{U}\}}$$

where $p = 1 - 2(r - d)/\sigma^2$. Similarly, there exists a static replication strategy for a up-and-out chained option UO_d with payoff at T :

$$\hat{f}^{UO_d}(S_T) = \left[f(S_T) + \left(\frac{S_T}{D} \right)^p f\left(\frac{D^2}{S_T}\right) \right] \mathbf{1}_{\{S_T < D\}} - \left[\left(\frac{S_T}{D} \right)^p f\left(\frac{D^2}{S_T}\right) + \left(\frac{D}{U} \right)^p f\left(\frac{U^2 S_T}{D^2}\right) \right] \mathbf{1}_{\{S_T < \frac{D^2}{U}\}}$$

3. Analysis of adjusted payoffs

In this section, the adjusted payoffs of DI_u and DO_u are analyzed. We first assume that the payoff $f(x)$ in Theorem 2.2 is given by the payoff of call, $f(x) = (x - K)^+$ at maturity $T = 1$. Then, the adjusted payoff of DI_u is

$$\begin{aligned} \hat{f}^{DI_u}(S_T) &= \left[\left(\frac{S_T}{U} \right)^p \left(\frac{U^2}{S_T} - K \right)^+ + \left(\frac{U}{D} \right)^p \left(\frac{D^2 S_T}{U^2} - K \right)^+ \right] \mathbf{1}_{\{S_T > \frac{U^2}{D}\}} \\ &= \left[\frac{K}{S_T} \left(\frac{S_T}{U} \right)^p \left(\frac{U^2}{K} - S_T \right)^+ + \left(\frac{U}{D} \right)^p \frac{D^2}{U^2} \left(S_T - \frac{KU^2}{D^2} \right)^+ \right] \mathbf{1}_{\{S_T > \frac{U^2}{D}\}} \end{aligned} \quad (3.1)$$

If $K \geq D$, then $\frac{U^2}{K} \leq \frac{U^2}{D}$ and the first term of (3.1) is zero. Thus,

$$\hat{f}^{DI_u}(S_T) = \left(\frac{U}{D} \right)^p \frac{D^2}{U^2} \left(S_T - \frac{KU^2}{D^2} \right)^+$$

In this case, $\hat{f}^{DI_u}(S_T)$ matches up to the payoff of a vanilla call option of volume $\left(\frac{U}{D} \right)^p \frac{D^2}{U^2}$ with strike price KU^2/D^2 . The left picture of Fig. 1 shows the graph of the adjusted payoff $\hat{f}^{DI_u}(S_T)$ under the assumption

that $K=100$, $U=102$, $D=98$, $r=0.05$, $d=0$, $\sigma=0.1$. After rebalancing at time t_1 when the upper barrier is hit, the adjusted payoff $\tilde{f}^{Dl_u}(S_T)$ in (2.3) is

$$\tilde{f}^{Dl_u}(S_T) = \left[(S_T - K)^+ + \left(\frac{S_T}{D} \right)^p \left(\frac{D^2}{S_T} - K \right)^+ \right] \mathbf{1}_{\{S_T < D\}} = \frac{K}{S_T} \left(\frac{S_T}{D} \right)^p \left(\frac{D^2}{K} - S_T \right)^+$$

The graph of $\tilde{f}^{Dl_u}(S_T)$ is a curve when $S_T < D^2/K$ because the function in front of the payoff of put option is a non-constant function of S_T . The right real line of Fig. 1 shows the graph of the adjusted payoff $\tilde{f}^{Dl_u}(S_T)$ under the assumption that $K=100$, $D=98$, $r=0.05$, $d=0$, $\sigma=0.1$. In order to hedge Dl_u when $K \geq D$, at initiation, one can purchase vanilla call options of volume $\left(\frac{U}{D}\right)^p \frac{D^2}{U^2}$ with strike price KU^2/D^2 , and if the upper barrier is hit, at time t_1 , sell the call options and buy the portfolio of European options that provides payoff $\tilde{f}^{Dl_u}(S_T)$. However, it may be difficult in practice to find a portfolio of finite European options that provides payoff $\tilde{f}^{Dl_u}(S_T)$, and one needs to make an approximation for infinite ones. In particular, if the interest rate equals the dividend rate $r=d$, then $p=1$ and

$$\tilde{f}^{Dl_u}(S_T) = \frac{K}{D} \left(\frac{D^2}{K} - S_T \right)^+$$

Then, one can exactly hedge Dl_u by buying vanilla put options of volume K/D with strike price D^2/K at time t_1 . The right dash line of Fig. 1 represents the graph of adjusted payoff $\tilde{f}^{Dl_u}(S_T)$ in this case under the assumption that $K=100$, $D=98$, $\sigma=0.1$ and $r=d$, respectively.

We now suppose that the payoff $f(x)$ in Theorem 2.3 is the payoff of call, $f(x) = (x - K)^+$. Then, using Eq. (3.1) the adjusted payoff of DO_u is written as

$$\hat{f}^{DO_u}(S_T) = \left[(S_T - K)^+ + \left(\frac{S_T}{U} \right)^p \left(\frac{U^2}{S_T} - K \right)^+ \right] \mathbf{1}_{\{S_T > U\}} \quad (3.1)$$

If $K < D$, then the adjusted payoff

$$\begin{aligned} \hat{f}^{DO_u}(S_T) &= (S_T - K) \mathbf{1}_{\{S_T > U\}} + \frac{K}{S_T} \left(\frac{S_T}{U} \right)^p \left(\frac{U^2}{K} - S_T \right) \mathbf{1}_{\{U < S_T < \frac{U^2}{K}\}} \quad (3.2) \\ &= (S_T - K) \mathbf{1}_{\{S_T > U\}} + \frac{K}{S_T} \left(\frac{S_T}{U} \right)^p \left(\frac{U^2}{K} - S_T \right) \mathbf{1}_{\{U < S_T < \frac{U^2}{D}\}} \\ &\quad - \left(\frac{U}{D} \right)^p \frac{D^2}{U^2} \left(S_T - \frac{KU^2}{D^2} \right) \mathbf{1}_{\{S_T > \frac{U^2}{D}\}} \end{aligned}$$

and the adjusted payoff $\tilde{f}^{DO_u}(S_T)$ at the time t_1 when the upper barrier is hit is

$$\begin{aligned} \tilde{f}^{DO_u}(S_T) &= (S_T - K)^+ \quad (3.3) \\ &= (S_T - K) \mathbf{1}_{\{S_T \geq D \text{ or } S_T \leq K\}} - \frac{K}{S_T} \left(\frac{S_T}{D} \right)^p \left(\frac{D^2}{K} - S_T \right) \mathbf{1}_{\{S_T < D\}}. \end{aligned}$$

We provide the graphs of the adjusted payoffs $\hat{f}^{DO_u}(S_T)$ and $\tilde{f}^{DO_u}(S_T)$ under the assumption that $K=95$, $U=102$, $D=98$, $r=0.05$, $d=0$, $\sigma=0.1$ in Fig. 2.

Fig. 3 shows the graphs of the adjusted payoffs $\hat{f}^{DO_u}(S_T)$ and $\tilde{f}^{DO_u}(S_T)$ when the payoff $f(x)$ is the payoff of put, $f(x) = (K - x)^+$. The parameter values that we used are $K=100$, $U=102$, $D=98$, $r=0.05$, $d=0$, $\sigma=0.1$.

4. Numerical results

In this section, we examine the hedging strategies proposed in Section 2. We compare the exact values of chained barrier option and the numerical values of vanilla options with adjusted payoff. The

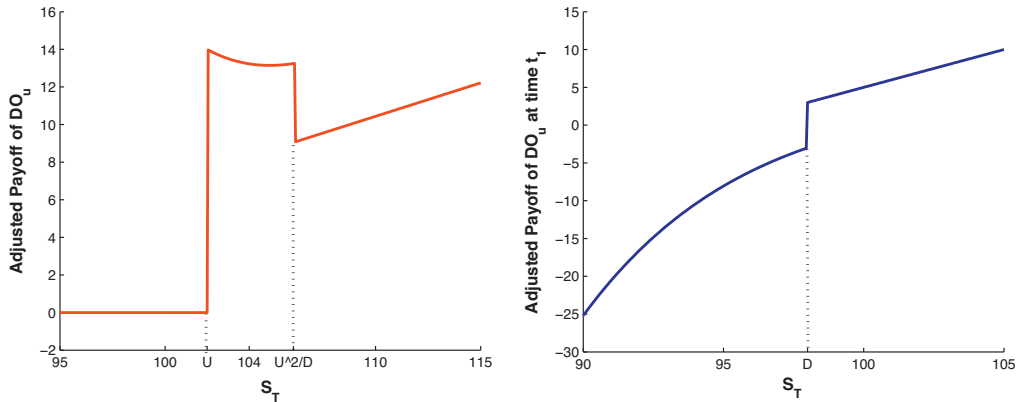


Fig. 2. Adjusted payoff of DO_u with $f(S_T) = (S_T - K)^+$ (option parameters: $K = 95$, $U = 102$, $D = 98$, $r = 0.05$, $d = 0$, $\sigma = 0.1$).

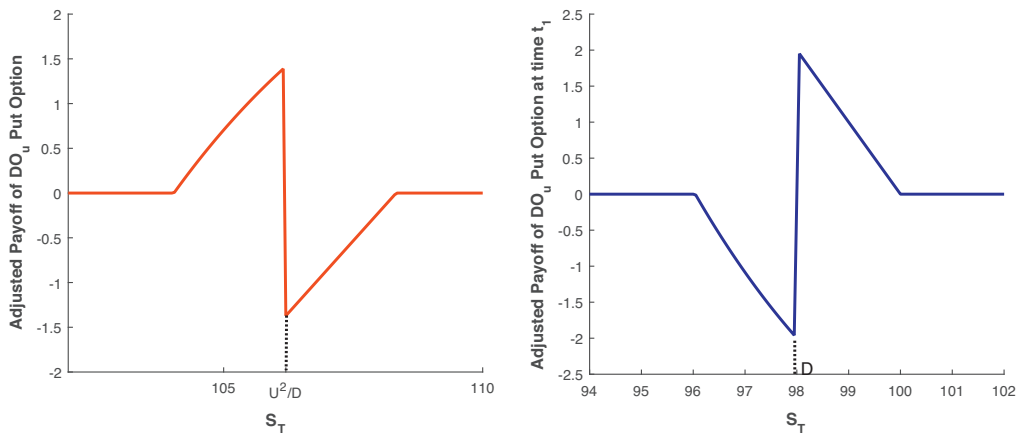


Fig. 3. Adjusted payoff of DO_u with $f(S_T) = (K - S_T)^+$ (option parameters: $K = 100$, $U = 102$, $D = 98$, $r = 0.05$, $d = 0$, $\sigma = 0.1$).

exact values are calculated from the closed formulae in Jun and Ku (2012), and the numerical values are obtained by Monte Carlo simulations.

We suppose that the payoff of barrier option at maturity T is $f(S_T) = (S_T - K)^+$. We then consider the values of down-and-in chained option DI_u and down-and-out chained option DO_u having payoff $f(S_T)$ at maturity $T = 1$ under the condition that upper barrier U was touched first.

Also, we simulate the values of down-and-in chained option DI_u and down-and-out chained option DO_u with adjusted payoffs at time 0. The prices of these options are given by

$$DI_u(t_0, S_0) = e^{-r(T-t_0)} E^Q[\hat{f}^{DI_u}(S_T)] \quad (4.1)$$

and

$$DO_u(t_0, S_0) = e^{-r(T-t_0)} E^Q[\hat{f}^{DO_u}(S_T)] \quad (4.2)$$

respectively. Let $t_0 = 0$ and the initial stock price $S_0 = 100$. $\hat{f}^{DI_u}(S_T)$ and $\hat{f}^{DO_u}(S_T)$ are given in Theorems 2.2 and 2.3. Then, we calculate (4.1) and (4.2) by using a variance reduction method called the Antithetic Variates method (see for example Glasserman, 2004). When we increase the number of sample paths for simulation from 10^4 to 10^7 , we see that the values of (4.1) and (4.2) converge to the exact option values in Fig. 4. When the option parameters $K = 100$, $U = 102$, $D = 98$, $r = 0.05$, $d = 0$ and $\sigma = 0.1$ are

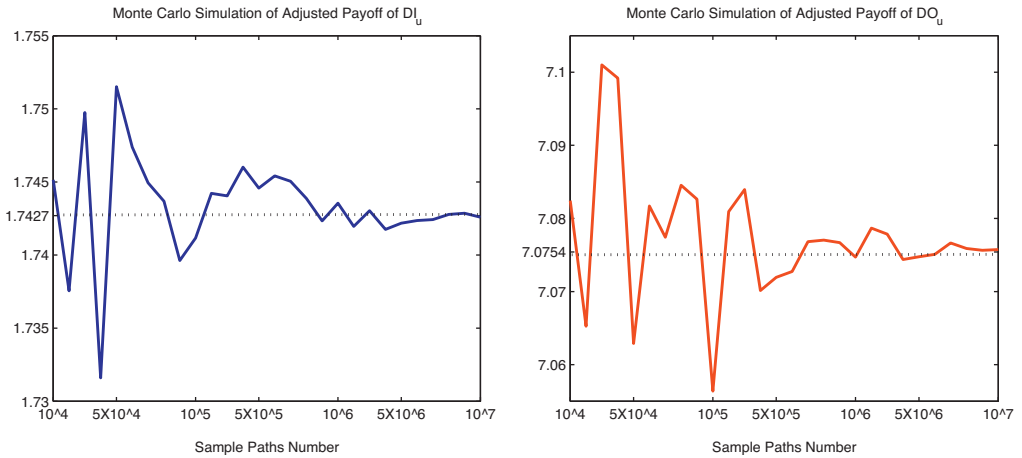


Fig. 4. Monte Carlo simulation of adjusted payoff of DI_u and DO_u with $f(S_T) = (S_T - K)^+$.

assumed, the exact value of down-and-in call DIC_u is 1.7427. Also, in the case that $K=95$, $U=102$, $D=98$, $r=0.05$, $d=0$ and $\sigma=0.1$, the exact value of down-and-out call DOC_u is 7.0754.

When K varies from 95 to 105, the exact values as well as the simulation values of DI_u and DO_u in the case of call option are presented in Table 1. Table 1 also shows the values of DI_u and DO_u when U and D vary. We observe that the simulation values using the adjusted payoffs are very close to the exact values of the options.

Table 1

Comparison of the exact values and Monte Carlo simulations for adjusted payoffs of DI_u and DO_u .

Strike	U	D	DI_u			DO_u		
			Exact	MC	Std	Exact	MC	Std
95	102	98	3.262572	3.262366	0.000637	7.075415	7.075289	0.000637
96			2.911261	2.911124	0.000497	6.671537	6.671219	0.000890
97			2.581468	2.581570	0.000967	6.267659	6.267532	0.000298
98			2.275792	2.275752	0.000768	5.863781	5.863487	0.000409
99			1.996280	1.996196	0.000627	5.460455	5.460359	0.000358
100			1.742672	1.742563	0.000484	5.060484	5.060455	0.000384
101			1.513949	1.513778	0.000494	4.667323	4.667320	0.000483
102			1.308905	1.308922	0.000368	4.284445	4.284449	0.000494
103			1.126188	1.126096	0.000490	3.915262	3.914939	0.000551
104			0.964332	0.964408	0.000438	3.563035	3.561363	0.000608
105			0.821798	0.821652	0.000246	3.230796	3.224414	0.000607
	101	95	3.262572	3.262366	0.000637	7.075415	7.075289	0.000637
		97	2.911261	2.911124	0.000497	6.671537	6.671219	0.000890
		99	2.581468	2.581570	0.000967	6.267659	6.267532	0.000298
	103	95	2.275792	2.275752	0.000768	5.863781	5.863487	0.000409
		97	1.996280	1.996196	0.000627	5.460455	5.460359	0.000358
		99	1.742672	1.742563	0.000484	5.060484	5.060455	0.000384
	105	95	1.513949	1.513778	0.000494	4.667323	4.667320	0.000483
		97	1.308905	1.308922	0.000368	4.284445	4.284449	0.000494
		99	1.126188	1.126096	0.000490	3.915262	3.914939	0.000551

Note. $f(S_T) = (S_T - K)^+$. Option parameters: $S_0 = 100$, $\sigma = 0.1$, $T = 1$, $r = 0.05$, $d = 0$. Sample paths number: 10^7 . Exact value is calculated from Jun and Ku (2012). MC: average of the discounted expectations of the adjusted payoff using the Antithetic Variates, a variance reduction method of Monte Carlo simulation. Std: standard deviation of the discounted expectations of the adjusted payoff.

Table 2
Delta hedging and static hedging for the chained option DI_u with varying up-barrier U , down-barrier D and volatility σ .

Hedge method	DI_u					
	$U = 103, D = 97$			$U = 105, D = 95$		
	Cost	Mean	Std	Cost	Mean	Std
Delta ($\sigma=0.2$)	4.4567	7.26%	36.44%	2.3351	17.72%	65.64%
Static ($\sigma=0.2$)	4.1686	0.33%	15.40%	1.9845	0.04%	21.11%
Delta ($\sigma=0.3$)	7.9409	−0.53%	25.29%	5.4707	6.35%	38.14%
Static ($\sigma=0.3$)	8.0286	0.57%	16.24%	5.1405	−0.07%	21.96%

Note. $f(S_T) = (S_T - K)^+$. Option parameters: $S_0 = 100, K = 100, T = 1, r = 0.05, d = 0$. Sample paths number: 10^4 . The exact option values are 4.1550 ($\sigma = 0.2$) and 7.9829 ($\sigma = 0.3$) for $U = 103, D = 97$, and 1.9836 ($\sigma = 0.2$) and 5.1442 ($\sigma = 0.3$) for $U = 105, D = 95$.

Table 2 shows a comparison of delta hedge and static hedge for DI_u with the payoff $f(S_T) = (S_T - K)^+$, and displays descriptive statistics for hedge errors for DI_u . Mean and Std are average and standard deviation of the hedging errors, respectively, relative to the exact option value of the chained call option DI_{C_u} . In the delta hedging method, hedging cost refers to the discounted total cost of delta hedging the option DI_u when daily rebalancing is adopted. In static hedging, hedging cost refers to the sum of initial cost (option value) and discounted hedging error where the hedging error is the difference between the replicating portfolio value and the option payoff at maturity T . We note that in the computation the static replicating portfolio contains five put options with different strike prices when the asset price hits the up-barrier U . In this computation, the strikes that we used are $K_i = A + 6 - 8i$ where $A = D^2/K$ for $i = 1, \dots, 5$. The weights of put options are obtained by solving the linear equations as in the following:

$$\begin{pmatrix} f(x_1) - \frac{K}{D}(A - x_1)^+ \\ f(x_2) - \frac{K}{D}(A - x_2)^+ \\ f(x_3) - \frac{K}{D}(A - x_3)^+ \\ f(x_4) - \frac{K}{D}(A - x_4)^+ \\ f(x_5) - \frac{K}{D}(A - x_5)^+ \end{pmatrix} = \begin{pmatrix} (K_1 - x_1)^+ & 0 & 0 & 0 & 0 \\ (K_1 - x_2)^+ & (K_2 - x_2)^+ & 0 & 0 & 0 \\ (K_1 - x_3)^+ & (K_2 - x_3)^+ & (K_3 - x_3)^+ & 0 & 0 \\ (K_1 - x_4)^+ & (K_2 - x_4)^+ & (K_3 - x_4)^+ & (K_4 - x_4)^+ & 0 \\ (K_1 - x_5)^+ & (K_2 - x_5)^+ & (K_3 - x_5)^+ & (K_4 - x_5)^+ & (K_5 - x_5)^+ \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix}$$

where α_i is the weight of i th put option with strike K_i and setting $x_i = A - 6i$. And then, the portfolio having five put options converts to 25 call options in the same way after the down-barrier D is hit. In Table 2 the static hedging method has the much lower standard deviations than the dynamic hedging method and static hedging is seen as an improvement over delta hedging.

5. Conclusion

This paper studies static hedging of barrier options where monitoring of the other barrier starts at a random time when the underlying asset price first crosses a certain barrier level. These chained-type barrier options have become popular in the over-the-counter equity and foreign exchange derivative markets, and the hedging problem is certainly an important research question. The hedging strategies are provided for a down-and-in chained option (DI_u) and a down-and-out chained option (DO_u) activated at the time when the underlying asset price hits an upper barrier level. The Monte Carlo simulation results for the values of vanilla options with adjusted payoffs are provided in order to examine the accuracy of the hedging strategies. Furthermore, a comparison between static hedging and delta hedging is shown to demonstrate static hedge performs better than dynamic hedge. A great advantage of the hedging strategies proposed in this paper is that they are applicable to barrier options with more complicated structure where more than two hitting times are chained together to activate barrier options.

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