

Analytic solution for American barrier options with two barriers



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ABSTRACT

This paper concerns American barrier options with two barriers. Standard American Options are difficult to price but there exist good numerical or analytical approximation methods. The situation is different for American barrier options. These options cease to exist or come into being if some price barrier is hit during the option's life. The paper studies analytic valuation of American barrier options with two barriers where the barriers become active by turns. In this paper, analytic valuation formulas for these options are derived by using both constant and exponential barriers for optimal early exercise policies.

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1. Introduction

Barrier options are widely used by institutional investors, banks and corporations in their risk management, and American-style options give their holders the additional flexibility of early exercise. Because a wide variety of traded options are American options, the problem of valuing American options has been an important topic in financial economics.

The first approach to valuing American options, proposed by Brennan and Schwartz [3] and Parkinson [26], was a direct numerical evaluation of the Black–Scholes partial differential equation using finite differences. Cox, Ross and Rubinstein [8] used the binomial model to reduce the size of errors by refining the time partition so that the resulting lattice has layers as close as possible to the barrier. These numerical methods are quite flexible and simple to implement. However, even after employing enhancement techniques such as control variates or convergence extrapolation, they are very time consuming.

There are many approximation schemes developed to reduce this time consuming task. Johnson [17] expressed the put value as an approximate function of its parameters. Geske and Johnson [11] approximated

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the American option price through an infinite series of multivariate normal distribution functions. Barone-Adesi and Whaley [1] used Merton's [25] solution for perpetual American options and the quadratic method of MacMillan [24]. Despite its high efficiency and the accuracy improvements, this method is not convergent because there is no control parameter to adjust to improve the accuracy.

Longstaff and Schwartz [23] adapted Monte Carlo simulation methods to deal with the American put problem. They addressed the optimal stopping problem in a Monte Carlo framework by comparing the conditional expected value of continuing with the value of immediate exercise if the option is currently in the money. Their method is based on a polynomial approximation of the continuation value, leading to an approximate free boundary. Sullivan [27] approximated the option value function through Chebyshev polynomials and employed a Gaussian quadrature integration scheme at each discrete exercise date. Although the speed and accuracy of the proposed numerical approximation can be enhanced via the Richardson extrapolation, its convergence properties are still unknown. Carr and Madan [6] showed how the fast Fourier Transform may be used to value options when the characteristic function of the return is known analytically. Belomestny et al. [2] proposed a novel approach to reduce the computational complexity of the dual method for pricing American options. They considered a sequence of martingales that converges to a given target martingale and decompose the original dual representation into a sum of representations that correspond to different levels of approximation to the target martingale.

Another stream of the American option pricing literature is to derive lower and upper bounds for American option values. Kim [22], Jacka [16], and Carr et al. [5] obtained an analytic integral-form solution for American options where the formulas represent the early premium of an American option as an integral which has the early exercise boundary. Broadie and Detemple [4] provided tight lower and upper bounds for American call prices based on the assumption that the early exercise boundary is a constant. Ju [18] approximated the early exercise boundary as a multipiece exponential function and substituted it by the early exercise premium integral, derived by Kim [22], to price American options. Ingersoll [14] described another approximation method of American options based on barrier derivatives: The exercise policy is approximated by a simple class of functions, and the best policy within that class is selected by standard optimization techniques. The advantages of this method are its simplicity and speed, even when used in general-purpose computer programs such as spreadsheets. Concretely, he dealt with a constant barrier approximation and an exponential barrier approximation for American put. Chung et al. [7] derived the essential formulas for solving the lower bound and the optimal exercise boundary.

For the American barrier option problem, Gao et al. [10] suggested an approximation method for American barrier options. They applied the approximation techniques of a standard American option to an American barrier option, and proposed two approximation methods using [13] and [18] to approximate an American barrier's exercise boundary. Dai and Kwok [9] provided an analytic formula for knock-in options and showed that the in-out barrier parity relationship for American barrier options could not be obtained unlike the case of European barrier options. Zhang et al. [28] presented the total least squares quasi-Monte Carlo approach for valuing American barrier options, which modifies the least-squares Monte Carlo method.

Ingersoll [14] presented American up-and-in put by an approximation method based on barrier options with constant exercise policies. He also suggested the use of an exponential function to approximate the early exercise boundary. Jun and Ku [21] provided analytic valuation formulas of American partial barrier options in terms of a finite sum of bivariate normal distribution functions.

To the best of our knowledge, there are no papers about American barrier options with two barriers. The main contribution of our study is that we are the first to study a barrier option of American type with two barriers and derive an analytic valuation formula. There are two classes of barrier options with two

barriers. One is double barrier options and the other is chained options.¹ Chained options differ from double barrier options in the sense that the other barrier does not exist until one barrier is hit according to their predetermined order. This paper investigates chained options of American type where an American barrier option is commenced at a time when the specified barrier is crossed before maturity. This paper extends the approximation method of American barrier option in Ingersoll [14] to the case of chained options of American type with two barriers, and derives the analytic formulas for such options. Consistent with intuition, when the first barrier of an American chained barrier option approaches the initial underlying asset price, the value of the American chained barrier option with two barriers converges to the American barrier option value with a single barrier (the second barrier of the two). Both constant and exponential functions are considered for early exercise boundaries in valuing chained option of American type. Our explicit formulas provide a very tight lower bound for the option price, and moreover, this method is superior in speed and its simplicity.

This article is organized as follows. Section 2 provides the valuation formulas for an American up-and-in put option which is activated at a first passage time to down-barrier D . This approximation method is based on barrier options along with constant early exercise policies. Section 3 treats the exponential exercise barriers case. Section 4 presents some conclusions.

2. Approximation of American chained option by constant barriers

Let r be the risk-free interest rate, q be a dividend rate, and $\sigma > 0$ be a constant. We assume the price of the underlying asset S follows a geometric Brownian motion

$$S_t = S_0 \exp(\mu t + \sigma W_t)$$

where $\mu = r - q - \frac{\sigma^2}{2}$ and W_t is a standard Brownian motion under the risk-neutral probability P .

In this section, we consider chained options of American type. American options give their holders the flexibility of early exercise. An American up-and-in put option can be exercised before the expiration time when it is in the money, but only after the stock price rises above the knock-in barrier. We consider the up-and-in put commencing at a time when the asset price hits the down-barrier D . This option can be exercised before the expiration time T by the option holder, but only after the underlying asset falls below D and then rises above U before time T . The payoff of this option is zero otherwise.

In order to obtain the approximation to valuing American chained option using barrier derivatives under exercise policies, it will be convenient to introduce the following digitals: Let $\mathcal{D}(S, t; A)$ be the value at time t of receiving one dollar at time T if and only if the event A occurs, and $\mathcal{DS}(S, t; A)$ be the value at time t of receiving one share of stock at time T if and only if the event A occurs. The \mathcal{D} is said to be a digital or binary option and the \mathcal{DS} is said to be a digital share. The quantity $\mathcal{E}(S, t, K_\tau; A)$ denotes the value at time t of payment $X - K_\tau$ at the first time τ that the stock price S hits the barrier K_τ provided the event A occurs before time T , where X is a strike price. The \mathcal{E} is said to be a first-touch digital.

First we present a brief review of the results in [14]. Consider an American up-and-in put expiring T with strike price X . Let us denote by U the up-barrier and by K_t^* the optimal exercise policy. Let τ_{B_1} denote the first time the stock price is equal to B_1 and $\tau_{B_1 B_2}$ denote the first time after τ_{B_1} that the stock price is equal to B_2 .

¹ For chained options, another barrier option is activated when a primary barrier is hit. For example, an up-and-in chained put option (UIP_d) is an up-and-in put option activated at a time when the underlying asset price hits a lower barrier level, D , and an up-and-in chained put option (UIP_{ud}) is an up-and-in put option which is activated at a time when the asset price crosses two different barrier levels (an up-barrier followed by a down-barrier). These options have become popular in the over-the-counter equity and foreign exchange derivative markets. We refer to Jun and Ku [19,20] for details.

Let $A_1 = \{t < \tau_U < T, \tau_{UK^*} > T, S_T < X\}$ be the event of exercise at maturity under the optimal policy, and $A_2 = \{t < \tau_U, \tau_{UK^*} < T\}$ be the event of early exercise under the optimal policy. Then the value of the up-and-in put can be written as

$$UIP = X \cdot \mathcal{D}(S, t; A_1) - \mathcal{D}\mathcal{S}(S, t; A_1) + \mathcal{E}(S, t, K_t^*; A_2).$$

The barrier approximation for this put takes the maximum value within a class of restricted policies. For example, for constant exercise policies k ,

$$UIP \geq UIP_{\text{const}} = \max_k [X \cdot \mathcal{D}(S, t; A'_1) - \mathcal{S}(S, t; A'_1) + \mathcal{E}(S, t, k; A'_2)]$$

where $A'_1 = \{t < \tau_U < T, \tau_{Uk} > T, S_T < X\}$, $A'_2 = \{t < \tau_U, \tau_{Uk} < T\}$, and τ_{Uk} is the first time the stock price hits the constant policy barrier k after hitting the barrier U .

The values for these digitals are given by

$$\begin{aligned} \mathcal{D}(S, t; A'_1) &= e^{-r(T-t)} \left\{ \left(\frac{U}{S_t} \right)^{\frac{2\mu}{\sigma^2}} \left[N \left(h_1 \left(\frac{U^2}{S_t k} \right) \right) - N \left(h_1 \left(\frac{U^2}{S_t X} \right) \right) \right] \right. \\ &\quad \left. + \left(\frac{k}{U} \right)^{\frac{2\mu}{\sigma^2}} \left[N \left(h_1 \left(\frac{S_t k^2}{U^2 X} \right) \right) - N \left(h_1 \left(\frac{S_t k}{U^2} \right) \right) \right] \right\} \\ \mathcal{D}\mathcal{S}(S, t; A'_1) &= S_t e^{-q(T-t)} \left\{ \left(\frac{U}{S_t} \right)^{\frac{2\bar{\mu}}{\sigma^2}} \left[N \left(h_2 \left(\frac{U^2}{S_t k} \right) \right) - N \left(h_2 \left(\frac{U^2}{S_t X} \right) \right) \right] \right. \\ &\quad \left. + \left(\frac{k}{U} \right)^{\frac{2\bar{\mu}}{\sigma^2}} \left[N \left(h_2 \left(\frac{S_t k^2}{U^2 X} \right) \right) - N \left(h_2 \left(\frac{S_t k}{U^2} \right) \right) \right] \right\} \\ \mathcal{E}(S, t, k; A'_2) &= (X - k) \left[\left(\frac{k}{S_t} \right)^{b-\beta} \left(\frac{k}{U} \right)^{2\beta} N \left(g_1 \left(\frac{S_t k}{U^2} \right) \right) \right. \\ &\quad \left. + \left(\frac{k}{S_t} \right)^{b+\beta} \left(\frac{U}{k} \right)^{2\beta} N \left(-g_1 \left(\frac{U^2}{S_t k} \right) \right) \right] \end{aligned}$$

where N is the standard normal distribution function,

$$\begin{aligned} h_1(z) &= \frac{\ln z + \mu(T-t)}{\sigma\sqrt{T-t}}, & h_2(z) &= \frac{\ln z + \bar{\mu}(T-t)}{\sigma\sqrt{T-t}}, & g_1(z) &= \frac{\ln z + \beta\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ \mu &= r - q - \frac{1}{2}\sigma^2, & \bar{\mu} &= r - q + \frac{1}{2}\sigma^2, & b &= \frac{\mu}{\sigma^2}, & \text{and } \beta &= \sqrt{b^2 + \frac{2r}{\sigma^2}}. \end{aligned}$$

Now, we present the valuation of American chained option using barrier derivatives. Let K^* denote the optimal exercise policy. We denote by τ_D the first time that the underlying asset reaches the barrier D . By τ_{DU} and τ_{DUK^*} , we denote the first time that the underlying asset rises to the barrier U after τ_D and the first time that the underlying asset falls to the optimal exercise policy K^* after τ_{DU} respectively. Let $A_3 = \{t < \tau_D, \tau_{DU} < T, \tau_{DUK^*} > T, S_T < X\}$ be the event of exercise at maturity under the optimal policy, and $A_4 = \{t < \tau_D, \tau_{DU}, \tau_{DUK^*} < T\}$ be the event of early exercise under the optimal policy.

Then the value of an American up-and-in put commencing at the first passage time to barrier D is written as

$$UIP_d = X\mathcal{D}(S, t; A_3) - \mathcal{D}\mathcal{S}(S, t; A_3) + \mathcal{E}(S, t, K_t^*; A_4).$$

For the barrier approximation of this option, we consider a class of all constant exercise policies. We let $A_5 = \{t < \tau_D, \tau_{DU} < T, \tau_{DUk} > T, S_T < X\}$ be the event of exercise at maturity under a constant policy k , and $A_6 = \{t < \tau_D, \tau_{DUk} < T\}$ be the event of early exercise under policy k . Then we can express the option price as

$$UIP_{d \text{ const}} = \max_{k \in \mathcal{K}_c} [X \cdot \mathcal{D}(S, t; A_5) - \mathcal{DS}(S, t; A_5) + \mathcal{E}(S, t, k; A_6)]. \tag{2.1}$$

If the set of policies considered contains all continuous functions, then the resulting put value will be exact. Since the set \mathcal{K}_c is the set of all constant functions, then the resulting value will be an approximation providing a (very tight) lower bound to the put price.

We first present the digital options for an American up-and-in put option starting at the hitting time of barrier D under constant exercise policies.

Theorem 2.1. *The values of a digital option and a digital share at time t for the event $A_5 = \{t < \tau_D, \tau_{DU} < T, \tau_{DUk} > T, S_T < X\}$ are*

$$\begin{aligned} \mathcal{D}(S, t; A_5) &= e^{-r(T-t)} \left(\frac{U}{D}\right)^{\frac{2\mu}{\sigma^2}} \left[N\left(h_1\left(\frac{U^2 S_t}{D^2 k}\right)\right) - N\left(h_1\left(\frac{U^2 S_t}{D^2 X}\right)\right) \right] \\ &\quad + e^{-r(T-t)} \left(\frac{Dk}{US_t}\right)^{\frac{2\mu}{\sigma^2}} \left[N\left(h_1\left(\frac{D^2 k^2}{U^2 S_t X}\right)\right) - N\left(h_1\left(\frac{D^2 k}{U^2 S_t}\right)\right) \right], \\ \mathcal{DS}(S, t; A_5) &= S_t e^{-q(T-t)} \left(\frac{U}{D}\right)^{\frac{2\bar{\mu}}{\sigma^2}} \left[N\left(h_2\left(\frac{U^2 S_t}{D^2 k}\right)\right) - N\left(h_2\left(\frac{U^2 S_t}{D^2 X}\right)\right) \right] \\ &\quad + S_t e^{-q(T-t)} \left(\frac{Dk}{US_t}\right)^{\frac{2\bar{\mu}}{\sigma^2}} \left[N\left(h_2\left(\frac{D^2 k^2}{U^2 S_t X}\right)\right) - N\left(h_2\left(\frac{D^2 k}{U^2 S_t}\right)\right) \right] \end{aligned}$$

where N is the standard normal distribution function,

$$h_1(z) = \frac{\ln z + \mu(T-t)}{\sigma\sqrt{T-t}}, \quad h_2(z) = \frac{\ln z + \bar{\mu}(T-t)}{\sigma\sqrt{T-t}}, \quad \text{and} \quad \bar{\mu} = r - q + \frac{1}{2}\sigma^2.$$

Proof. Apply Lemma A.3 with letting $u = \frac{1}{\sigma} \ln \frac{U}{S_t}$, $d = \frac{1}{\sigma} \ln \frac{D}{S_t}$, $l = \frac{1}{\sigma} \ln \frac{k}{S_t}$, and $x = \frac{1}{\sigma} \ln \frac{X}{S_t}$ to derive the risk-neutral probability of exercise at maturity. We note that $N(x) - N(y) = N(-y) - N(-x)$. Then

$$\begin{aligned} &P(t < \tau_D, \tau_{DU} < T, \tau_{DUk} > T, S_T < X \mid S_t) \\ &= \left(\frac{U}{D}\right)^{\frac{2\mu}{\sigma^2}} \left[N\left(h_1\left(\frac{U^2 S_t}{D^2 k}\right)\right) - N\left(h_1\left(\frac{U^2 S_t}{D^2 X}\right)\right) \right] \\ &\quad + \left(\frac{Dk}{US_t}\right)^{\frac{2\mu}{\sigma^2}} \left[N\left(h_1\left(\frac{D^2 k^2}{U^2 S_t X}\right)\right) - N\left(h_1\left(\frac{D^2 k}{U^2 S_t}\right)\right) \right]. \end{aligned}$$

Then the value of the digital option $\mathcal{D}(S, t; A_5)$ at time t is

$$\mathcal{D}(S, t; A_5) = e^{-r(T-t)} P(t < \tau_D, \tau_{DU} < T, \tau_{DUk} > T, S_T < X).$$

Also, the digital share $\mathcal{DS}(S, t; A_5)$ can be valued by changing μ to $\bar{\mu} = r - q + \frac{1}{2}\sigma^2$ in h_1 and replacing the discount factor $e^{-r(T-t)}$ by $S_t e^{-q(T-t)}$. (See for example [15].) \square

Theorem 2.2. *The value of the first-touch digital for the event A_6 is*

$$\begin{aligned} \mathcal{E}(S, t, k; A_6) &= (X - k) \left[\left(\frac{D}{U} \right)^{\beta-b} \left(\frac{Dk}{US_t} \right)^{\beta+b} N \left(g_1 \left(\frac{D^2k}{U^2S_t} \right) \right) \right. \\ &\quad \left. + \left(\frac{US_t}{Dk} \right)^{\beta-b} \left(\frac{U}{D} \right)^{\beta+b} N \left(-g_1 \left(\frac{U^2S_t}{D^2k} \right) \right) \right] \end{aligned}$$

where

$$g_1(z) = \frac{\ln z + \beta\sigma^2(T - t)}{\sigma\sqrt{T - t}}.$$

Proof. When the stock pays dividends, the asset price follows the continuous diffusion process $dS_t = (r - q)S_t dt + \sigma S_t dW_t$. To eliminate the dividend term in the process, we set

$$V_t = S_t^{\beta-b}$$

where

$$b = \frac{\mu}{\sigma^2} \quad \text{and} \quad \beta = \sqrt{b^2 + \frac{2r}{\sigma^2}}. \tag{2.2}$$

Then, by Ito’s lemma,

$$dV_t = rV_t dt + (\beta - b)\sigma V_t dW_t. \tag{2.3}$$

We may apply Lemma A.5 to the process V_t , since (2.3) does not contain the dividend term. The barriers for V_t corresponding to U , D , and k are $U^{\beta-b}$, $D^{\beta-b}$, and $k^{\beta-b}$ respectively. Furthermore, the volatility σ is replaced by $(\beta - b)\sigma$. Then, the value of the first-touch digital for the event A_6 is

$$\begin{aligned} \mathcal{E}(V, t, k^{\beta-b}; A_6) &= \frac{X - k}{k^{\beta-b}} V_t \left[\left(\frac{D^{\beta-b} k^{\beta-b}}{U^{\beta-b} V_t} \right)^{\frac{2r}{(\beta-b)^2\sigma^2} + 1} N \left(g_1 \left(\frac{D^2k}{U^2 V_t^{\frac{1}{\beta-b}}} \right) \right) \right. \\ &\quad \left. + \left(\frac{U^{\beta-b}}{D^{\beta-b}} \right)^{\frac{2r}{(\beta-b)^2\sigma^2} + 1} N \left(-g_1 \left(\frac{U^2 V_t^{\frac{1}{\beta-b}}}{D^2k} \right) \right) \right] \end{aligned}$$

where

$$g_1(z) = \frac{\ln z + \beta\sigma^2(T - t)}{\sigma\sqrt{T - t}}.$$

Thus,

$$\begin{aligned} \mathcal{E}(S, t, k; A_6) &= (X - k) \left(\frac{S_t}{k} \right)^{\beta-b} \left[\left(\frac{Dk}{US_t} \right)^{(\beta-b)\left(\frac{2r}{(\beta-b)^2\sigma^2} + 1\right)} N \left(g_1 \left(\frac{D^2k}{U^2S_t} \right) \right) \right. \\ &\quad \left. + \left(\frac{U}{D} \right)^{(\beta-b)\left(\frac{2r}{(\beta-b)^2\sigma^2} + 1\right)} N \left(-g_1 \left(\frac{U^2S_t}{D^2k} \right) \right) \right] \\ &= (X - k) \left[\left(\frac{D}{U} \right)^{\beta-b} \left(\frac{Dk}{US_t} \right)^{\beta+b} N \left(g_1 \left(\frac{D^2k}{U^2S_t} \right) \right) \right. \\ &\quad \left. + \left(\frac{US_t}{Dk} \right)^{\beta-b} \left(\frac{U}{D} \right)^{\beta+b} N \left(-g_1 \left(\frac{U^2S_t}{D^2k} \right) \right) \right]. \quad \square \end{aligned}$$

Table 1
Comparison of American chained put option values $UIP_{d\text{const}}$ with varying S_0 and strike price X .

S_0	X	$V(10)$	$V(30)$	$V(50)$	$V(100)$	$V(500)$	MC	k^*
96	95	1.5138	1.5155	1.5161	1.5161	1.5162	1.5188	78.22
	97.5	1.9478	1.9486	1.9503	1.9503	1.9503	1.9699	79.80
	100	2.4647	2.4647	2.4670	2.4670	2.4671	2.4400	81.72
	102.5	3.0712	3.0712	3.0729	3.0732	3.0734	3.0531	83.44
	105	3.7730	3.7730	3.7731	3.7747	3.7747	3.7686	85.14
98	95	1.2554	1.2573	1.2574	1.2575	1.2576	1.2600	78.46
	97.5	1.6292	1.6309	1.6316	1.6316	1.6317	1.6418	80.22
	100	2.0787	2.0793	2.0812	2.0812	2.0812	2.0604	81.97
	102.5	2.6107	2.6107	2.6130	2.6130	2.6132	2.6260	83.70
	105	3.2315	3.2315	3.2330	3.2335	3.2336	3.2050	85.43
100	95	1.0379	1.0399	1.0397	1.0400	1.0400	1.0522	78.70
	97.5	1.3587	1.3606	1.3609	1.3609	1.3610	1.3698	80.46
	100	1.7478	1.7493	1.7503	1.7503	1.7503	1.7632	82.22
	102.5	2.2125	2.2128	2.2150	2.2150	2.2150	2.2426	83.97
	105	2.7591	2.7591	2.7614	2.7614	2.7616	2.7339	85.70
102	95	0.8557	0.8575	0.8573	0.8576	0.8576	0.8714	78.90
	97.5	1.1298	1.1318	1.1317	1.1319	1.1320	1.1463	80.68
	100	1.4653	1.4672	1.4676	1.4676	1.4677	1.4624	82.46
	102.5	1.8694	1.8708	1.8720	1.8720	1.8720	1.8939	84.21
	105	2.3488	2.3489	2.3514	2.3514	2.3515	2.3535	85.95
104	95	0.7036	0.7053	0.7051	0.7052	0.7053	0.7164	79.10
	97.5	0.9369	0.9388	0.9386	0.9389	0.9389	0.9482	80.90
	100	1.2251	1.2271	1.2271	1.2273	1.2273	1.2218	82.68
	102.5	1.5751	1.5770	1.5776	1.5776	1.5777	1.5634	84.45
	105	1.9939	1.9951	1.9966	1.9966	1.9966	2.0296	86.21

Option parameters: $U = 105, D = 95, \sigma = 0.3, T = 0.5, r = 0.05, q = 0$. $V(N)$ is an option value of $UIP_{d\text{const}}$ using the formulas in (2.1). N is the number of constant policy barriers which are evenly spaced from 0 to 100. MC is a result of simulation using the Antithetic Variates, a Variance Reduction Method of Monte Carlo simulation. k^* is the optimal policy barrier for $V(10\,000)$.

We next present some numerical results to examine the accuracy of our solution. We compute the values of American chained option by our formula (2.1) and compare them with those by popular Monte Carlo method with an Antithetic Variate. (See for example [12].)

Table 1 shows the values of American up-and-in put option which is activated at time when barrier D is hit with varying initial price S_0 and strike price X . The parameter values that we used are $U = 105, D = 95, \sigma = 0.3, T = 0.5, r = 0.05, q = 0$. The values of S_0 vary from 96 to 104 and the values of X from 95 to 105. Table 2 shows the values of American up-and-in put option with varying levels of upper barrier U and down barrier D . The parameter values used are $S_0 = 100, X = 100, \sigma = 0.3, T = 0.5, r = 0.05, q = 0$. The values of U vary from 101 to 109 and the values of D from 91 to 99.

$V(N)$ is an option value of $UIP_{d\text{const}}$ using barrier options with constant policy barriers in (2.1). N is the element number of constant policy set \mathcal{K}_c to seek the best policy where policies are evenly spaced from 0 to $X = 100$. Since the American up-and-in put option comes into action only if the down-barrier is hit first, the option price $UIP_{d\text{const}}$ decreases as the initial stock price gets farther apart from the down-barrier D . We notice that as the element number N of constant policy set increases, the option value $V(N)$ increases concavely. In other words, it converges to a constant quickly as N becomes large.

MC is a result of simulation using the Antithetic Variates, a Variance Reduction Method of Monte Carlo simulation. For the American chained barrier option using policy barriers, Monte Carlo method requires much larger amount of computer time because a large number of sample paths and policy barriers, and a large enough monitoring frequency must be needed in order to catch the hitting times. For the Monte Carlo approximation in Table 1, the computer time is more than 10 000 times as long as for our formulas method to obtain the similar results under the same policy numbers. For the MC results in Table 1 and Table 2, a monitoring frequency is 1000, the number of sample paths is 1000, and the number of policy barriers (evenly spaced from 0 to 100) is 100.

Table 2
Comparison of American chained put option values $UIP_{d \text{ const}}$ with varying U and D .

U	D	$V(10)$	$V(30)$	$V(50)$	$V(100)$	$V(500)$	MC	k^*
101	91	1.6245	1.6263	1.6270	1.6270	1.6270	1.6456	82.32
	93	2.3484	2.3484	2.3508	2.3508	2.3508	2.3232	81.79
	95	3.2735	3.2735	3.2745	3.2754	3.2754	3.2221	81.24
	97	4.4141	4.4141	4.4141	4.4148	4.4149	4.3007	80.69
	99	5.7743	5.7743	5.7743	5.7743	5.7743	5.5791	80.12
103	91	1.1351	1.1371	1.1371	1.1373	1.1373	1.1600	82.78
	93	1.6865	1.6882	1.6890	1.6890	1.6891	1.6780	82.27
	95	2.4128	2.4128	2.4152	2.4152	2.4152	2.4029	81.75
	97	3.3340	3.3340	3.3348	3.3358	3.3358	3.2997	81.21
	99	4.4628	4.4628	4.4628	4.4634	4.4636	4.4202	80.67
105	91	0.7790	0.7808	0.7806	0.7808	0.7808	0.8034	83.21
	93	1.1900	1.1921	1.1921	1.1922	1.1923	1.1694	82.72
	95	1.7478	1.7493	1.7503	1.7503	1.7503	1.7632	82.22
	97	2.4759	2.4759	2.4782	2.4782	2.4783	2.4800	81.71
	99	3.3928	3.3928	3.3935	3.3946	3.3946	3.4294	81.18
107	91	0.5254	0.5268	0.5267	0.5267	0.5268	0.5506	83.62
	93	0.8254	0.8272	0.8270	0.8272	0.8272	0.8389	83.15
	95	1.2448	1.2469	1.2469	1.2471	1.2471	1.2648	82.67
	97	1.8083	1.8097	1.8108	1.8108	1.8108	1.8144	82.17
	99	2.5377	2.5377	2.5400	2.5400	2.5401	2.5248	81.67
109	91	0.3485	0.3494	0.3494	0.3494	0.3494	0.3802	84.01
	93	0.5629	0.5644	0.5643	0.5643	0.5644	0.5937	83.55
	95	0.8720	0.8739	0.8736	0.8739	0.8739	0.9034	83.09
	97	1.2994	1.3014	1.3015	1.3016	1.3017	1.3057	82.61
	99	1.8680	1.8693	1.8706	1.8706	1.8706	1.8706	82.13

Option parameters: $S_0 = 100$, $X = 100$, $\sigma = 0.3$, $T = 0.5$, $r = 0.05$, $q = 0$. $V(N)$ is an option value of $UIP_{d \text{ const}}$ using the formulas in (2.1). N is the number of constant policy barriers which are evenly spaced from 0 to 100. MC is a result of simulation using the Antithetic Variates, a Variance Reduction Method of Monte Carlo simulation. k^* is the optimal policy barrier for $V(10\,000)$.

We note that the last column k^* is the optimal policy barrier when $N = 10\,000$, and the best constant policy depends, of course, on option parameters such as initial stock price, strike price, upper barrier, and lower barrier.

Remark 2.3. When the barrier D approaches to the current stock price S_t , it can be checked that the formulas for \mathcal{D} , \mathcal{DS} and \mathcal{E} become the values of these digitals for the regular American barrier option given in this section.

3. Approximation of American chained option by exponential barriers

In this section, we present the valuation formulas to approximate an American chained option under exponential policy barriers. The barrier approximation method can be improved by using an expansive class of functions for exercise policies. We consider the class of exercise policies, \mathcal{K}_e , is a set of exponential functions whose elements are in the form of $K_t = K_0 e^{\delta t}$ with constant K_0 and $\delta \geq 0$. Since options with exponential barriers have analytical solutions under Black–Scholes conditions, an exponential barrier is a natural choice. Let $A_7 = \{t < \tau_D, \tau_{DU} < T, \tau_{DUK_t} > T, S_T < X\}$ be the event of exercise at maturity under an exponential policy barrier K_t and $A_8 = \{t < \tau_D, \tau_{DUK_t} < T\}$ be the event of early exercise under policy K_t . Then, the American up-and-in put option activated at time when barrier D is hit is approximated under exponential policy barriers as

$$UIP_{d \text{ exp}} = \max_{K_t \in \mathcal{K}_e} [X \cdot \mathcal{D}(S, t; A_7) - \mathcal{DS}(S, t; A_7) + \mathcal{E}(S, t, K_t; A_8)].$$

Theorem 3.1. The values of a digital option and a digital share at time t for the event $A_7 = \{t < \tau_D, \tau_{DU} < T, \tau_{DUK_t} > T, S_T < X\}$ are

$$\begin{aligned} \mathcal{D}(S, t; A_7) &= e^{-r(T-t)} \left(\frac{U}{D} \right)^{\frac{2\mu}{\sigma^2}} \left[N \left(h_4 \left(\frac{U^2 S_t}{D^2 K_t} \right) \right) - N \left(h_1 \left(\frac{U^2 S_t}{D^2 X} \right) \right) \right] \\ &\quad + e^{-r(T-t)} \left(\frac{DK_t}{US_t} \right)^{\frac{2\mu}{\sigma^2}} \left(\frac{U^2 S_t}{D^2 K_t} \right)^{\frac{2\delta}{\sigma^2}} \left[N \left(h_1 \left(\frac{D^2 K_t^2}{U^2 S_t X} \right) \right) - N \left(h_4 \left(\frac{D^2 K_t}{U^2 S_t} \right) \right) \right], \\ \mathcal{DS}(S, t; A_7) &= S_t e^{-q(T-t)} \left(\frac{U}{D} \right)^{\frac{2\bar{\mu}}{\sigma^2}} \left[N \left(h_5 \left(\frac{U^2 S_t}{D^2 K_t} \right) \right) - N \left(h_2 \left(\frac{U^2 S_t}{D^2 X} \right) \right) \right] \\ &\quad + S_t e^{-q(T-t)} \left(\frac{DK_t}{US_t} \right)^{\frac{2\bar{\mu}}{\sigma^2}} \left(\frac{U^2 S_t}{D^2 K_t} \right)^{\frac{2\delta}{\sigma^2}} \left[N \left(h_2 \left(\frac{D^2 K_t^2}{U^2 S_t X} \right) \right) - N \left(h_5 \left(\frac{D^2 K_t}{U^2 S_t} \right) \right) \right] \end{aligned}$$

where

$$h_4(z) = \frac{\ln z + (\mu - \delta)(T - t)}{\sigma \sqrt{T - t}} \quad \text{and} \quad h_5(z) = \frac{\ln z + (\bar{\mu} - \delta)(T - t)}{\sigma \sqrt{T - t}}.$$

Remark 3.2. If the barrier grows exponentially at rate δ , the drift of S_t , μ , and the strike price X can be considered as $\mu - \delta$ and $Xe^{-\delta T}$ respectively, for a constant barrier. Therefore, the function h_4 appears in the above formula, and the terms which depend on X still use h_1 since the changing barrier effects cancel. However, we note that for chained options, exponential barrier appears only after both barriers U and D are hit, and thus just replacing the drift does not lead to the formulas in [Theorem 3.1](#).

Proof of Theorem 3.1. Let $k_t = k_0 + \frac{\delta}{\sigma} t$ where k_0 and $\delta \geq 0$ are constants. We first consider the probability that the process X_t falls below d , and then rises above u , and then does not hit k_t before time T , and its value at time T is less than x .

$$\begin{aligned} &P(\tau_{du} < T, \tau_{duk_t} > T, X_T \leq x \mid X_0 = 0) \\ &= P(\tau_{du} < T, X_T \leq x \mid X_0 = 0) - P(\tau_{duk_t} \leq T \mid X_0 = 0) \\ &\quad + P(\tau_{duk_t} \leq T, X_T > x \mid X_0 = 0) \end{aligned} \tag{3.1}$$

The last term in [\(3.1\)](#) can be calculated as in the proof of [Lemma A.1](#). Since d and u are constants,

$$\begin{aligned} &P(\tau_{duk_t} \leq T, X_T > x \mid X_0 = 0) \\ &= \exp\left(\frac{2\mu}{\sigma}(u - d)\right) P(\tau_{k_t} \leq T, X_T > x \mid X_0 = 2u - 2d). \end{aligned}$$

Let $X_t - \frac{\delta}{\sigma} t = \tilde{X}_t$. Then the drift of \tilde{X}_t is $\frac{\mu - \delta}{\sigma}$ and

$$\begin{aligned} &P(\tau_{k_t} \leq T, X_T > x \mid X_0 = 2u - 2d) \\ &= P\left(\tilde{\tau}_{k_0} \leq T, \tilde{X}_T > x - \frac{\delta}{\sigma} T \mid \tilde{X}_0 = 2u - 2d\right) \\ &= \exp\left(\frac{2(\mu - \delta)}{\sigma}(k_0 - 2u + 2d)\right) P\left(\tilde{X}_T > x - \frac{\delta}{\sigma} T \mid \tilde{X}_0 = 2k_0 - 2u + 2d\right) \\ &= \exp\left(\frac{2(\mu - \delta)}{\sigma}(k_0 - 2u + 2d)\right) N\left(\frac{2k_0 - 2u + 2d - x + \frac{\mu}{\sigma} T}{\sqrt{T}}\right). \end{aligned}$$

Thus,

$$\begin{aligned}
 &P(\tau_{duk_t} \leq T, X_T > x \mid X_0 = 0) \\
 &= \exp\left(\frac{2\mu}{\sigma}(k_0 - u + d)\right) \exp\left(\frac{2\delta}{\sigma}(2u - 2d - k_0)\right) N\left(\frac{2k_0 - 2u + 2d - x + \frac{\mu}{\sigma}T}{\sqrt{T}}\right). \tag{3.2}
 \end{aligned}$$

The first two terms in (3.1) can be calculated by using the proof of Lemma A.2 for the process \tilde{X}_t . Therefore,

$$\begin{aligned}
 &P(\tau_{du} < T, \tau_{duk_t} > T, X_T \leq x \mid X_0 = 0) \\
 &= \exp\left(\frac{2\mu}{\sigma}(u - d)\right) \left[N\left(\frac{x - 2u + 2d - \frac{\mu}{\sigma}T}{\sqrt{T}}\right) - N\left(\frac{k_0 - 2u + 2d - (\frac{\mu - \delta}{\sigma})T}{\sqrt{T}}\right) \right] \\
 &\quad + \exp\left(\frac{2\mu}{\sigma}(k_0 - u + d)\right) \exp\left(\frac{2\delta}{\sigma}(2u - 2d - k_0)\right) \\
 &\quad \times \left[N\left(\frac{2k_0 - 2u + 2d - x + \frac{\mu}{\sigma}T}{\sqrt{T}}\right) - N\left(\frac{k_0 - 2u + 2d + (\frac{\mu - \delta}{\sigma})T}{\sqrt{T}}\right) \right].
 \end{aligned}$$

We set $u = \frac{1}{\sigma} \ln \frac{U}{S_t}$, $d = \frac{1}{\sigma} \ln \frac{D}{S_t}$, $k_t = \frac{1}{\sigma} \ln \frac{K_t}{S_t} = \frac{1}{\sigma} \ln(\frac{K_0 e^{\delta t}}{S_t})$, and $x = \frac{1}{\sigma} \ln \frac{X}{S_t}$ to derive the risk-neutral probability of exercise at maturity.

$$\begin{aligned}
 &P(t < \tau_{DU} < T, \tau_{DUK_t} > T, S_T < X \mid S_t) \\
 &= \left(\frac{U}{D}\right)^{\frac{2\mu}{\sigma^2}} \left[N\left(h_4\left(\frac{U^2 S_t}{D^2 K_t}\right)\right) - N\left(h_1\left(\frac{U^2 S_t}{D^2 X}\right)\right) \right] \\
 &\quad + \left(\frac{DK_t}{US_t}\right)^{\frac{2\mu}{\sigma^2}} \left(\frac{U^2 S_t}{D^2 K_t}\right)^{\frac{2\delta}{\sigma^2}} \left[N\left(h_1\left(\frac{D^2 K_t^2}{U^2 S_t X}\right)\right) - N\left(h_4\left(\frac{D^2 K_t}{U^2 S_t}\right)\right) \right]
 \end{aligned}$$

where

$$h_4(z) = \frac{\ln z + (\mu - \delta)(T - t)}{\sigma\sqrt{T - t}}.$$

As a result, the values of $\mathcal{D}(S, t; A_7)$ and $\mathcal{DS}(S, t; A_7)$ are obtained. \square

Theorem 3.3. *The value of a first-touch digital for the event $A_8 = \{t < \tau_D, \tau_{DUK_t} < T\}$ is*

$$\begin{aligned}
 \mathcal{E}(S, t, K_t; A_8) &= X \left[\left(\frac{D}{U}\right)^{\beta - b} \left(\frac{DK_t}{US_t}\right)^{\beta + b} \left(\frac{U^2 S_t}{D^2 K_t}\right)^{\frac{2\delta}{(\beta - b)\sigma^2}} N\left(g_2\left(\frac{D^2 K_t}{U^2 S_t}\right)\right) \right. \\
 &\quad \left. + \left(\frac{US_t}{DK_t}\right)^{\beta - b} \left(\frac{U}{D}\right)^{\beta + b} N\left(-g_2\left(\frac{U^2 S_t}{D^2 K_t}\right)\right) \right] \\
 &\quad - K_t \left[\left(\frac{D}{U}\right)^{\beta_1 - b_1} \left(\frac{DK_t}{US_t}\right)^{\beta_1 + b_1} \left(\frac{U^2 S_t}{D^2 K_t}\right)^{\frac{2\delta}{(\beta_1 - b_1)\sigma^2}} N\left(g_3\left(\frac{D^2 K_t}{U^2 S_t}\right)\right) \right. \\
 &\quad \left. + \left(\frac{US_t}{DK_t}\right)^{\beta_1 - b_1} \left(\frac{U}{D}\right)^{\beta_1 + b_1} N\left(-g_3\left(\frac{U^2 S_t}{D^2 K_t}\right)\right) \right]
 \end{aligned}$$

where

$$b_1 = \frac{\mu - \delta}{\sigma^2}, \quad \beta_1 = \sqrt{b_1^2 + \frac{2(r - \delta)}{\sigma^2}}, \quad g_2(z) = \frac{\ln z + (\beta\sigma^2 - \frac{\delta}{\beta - b})(T - t)}{\sigma\sqrt{T - t}}, \quad \text{and}$$

$$g_3(z) = \frac{\ln z + (\beta_1\sigma^2 - \frac{\delta}{\beta_1 - b_1})(T - t)}{\sigma\sqrt{T - t}}.$$

Proof. We consider the first-touch digital with a barrier of $K_0e^{\delta t}$ and a barrier payment of $X - K_0e^{\delta t}$ at hitting time.

$$P(\tau_{duk_t} \leq T \mid X_0 = 0) = P(\tau_{duk_t} \leq T, X_T > k_T \mid X_0 = 0) + P(\tau_{duk_t} \leq T, X_T \leq k_T \mid X_0 = 0) \tag{3.3}$$

where $k_t = k_0 + \frac{\delta}{\sigma}t$ with constant k_0 and $\delta \geq 0$. The first term in (3.3) is obtained from (3.2), i.e.,

$$P(\tau_{duk_t} \leq T, X_T > k_T \mid X_0 = 0) = \exp\left(\frac{2\mu}{\sigma}(k_0 - u + d)\right) \exp\left(\frac{2\delta}{\sigma}(2u - 2d - k_0)\right) N\left(\frac{k_0 - 2u + 2d + (\frac{\mu - \delta}{\sigma})T}{\sqrt{T}}\right).$$

Also, the second term in (3.3) is computed as

$$P(\tau_{duk_t} \leq T, X_T \leq k_T \mid X_0 = 0) = P(\tau_{du} \leq T, X_T \leq k_T \mid X_0 = 0) = \exp\left(\frac{2\mu}{\sigma}(u - d)\right) P(X_T \leq k_T \mid X_0 = 2u - 2d) = \exp\left(\frac{2\mu}{\sigma}(u - d)\right) N\left(\frac{k_0 - 2u + 2d - (\frac{\mu - \delta}{\sigma})T}{\sqrt{T}}\right).$$

Therefore

$$P(\tau_{duk_t} \leq T \mid X_0 = 0) = \exp\left(\frac{2\mu}{\sigma}(u - d)\right) N\left(\frac{k_0 - 2u + 2d - (\frac{\mu - \delta}{\sigma})T}{\sqrt{T}}\right) + \exp\left(\frac{2\mu}{\sigma}(k_0 - u + d)\right) \exp\left(\frac{2\delta}{\sigma}(2u - 2d - k_0)\right) N\left(\frac{k_0 - 2u + 2d + (\frac{\mu - \delta}{\sigma})T}{\sqrt{T}}\right).$$

Let $u = \frac{1}{\sigma} \ln \frac{U}{S_t}$, $d = \frac{1}{\sigma} \ln \frac{D}{S_t}$, $k_t = \frac{1}{\sigma} \ln \frac{K_t}{S_t} = \frac{1}{\sigma} \ln(\frac{K_0e^{\delta t}}{S_t})$, and $x = \frac{1}{\sigma} \ln \frac{X}{S_t}$. Then the substitution leads to

$$P(t < \tau_D, \tau_{DUK_t} \leq T \mid S_t) = \left(\frac{U}{D}\right)^{\frac{2\mu}{\sigma^2}} N\left(-h_4\left(\frac{U^2 S_t}{D^2 K_t}\right)\right) + \left(\frac{DK_t}{US_t}\right)^{\frac{2\mu}{\sigma^2}} \left(\frac{U^2 S_t}{D^2 K_t}\right)^{\frac{2\delta}{\sigma^2}} N\left(h_4\left(\frac{D^2 K_t}{U^2 S_t}\right)\right).$$

The value of digital share $\mathcal{DS}(S, t; A_8)$ at time t is

$$\mathcal{DS}(S, t; A_8) = S_t e^{-q(T-t)} \left[\left(\frac{U}{D}\right)^{\frac{2\mu}{\sigma^2}} N\left(-h_5\left(\frac{U^2 S_t}{D^2 K_t}\right)\right) + \left(\frac{DK_t}{US_t}\right)^{\frac{2\mu}{\sigma^2}} \left(\frac{U^2 S_t}{D^2 K_t}\right)^{\frac{2\delta}{\sigma^2}} N\left(h_5\left(\frac{D^2 K_t}{U^2 S_t}\right)\right) \right]$$

where

$$h_5(z) = \frac{\ln z + (\bar{\mu} - \delta)(T - t)}{\sigma\sqrt{T - t}}.$$

If the stock does not pay dividends,

$$\begin{aligned} \mathcal{D}\mathcal{S}(S, t; A_8) &= S_t \left[\left(\frac{U}{D} \right)^{\frac{2r}{\sigma^2} + 1} N \left(-h_6 \left(\frac{U^2 S_t}{D^2 K_t} \right) \right) + \left(\frac{DK_t}{US_t} \right)^{\frac{2r}{\sigma^2} + 1} \left(\frac{U^2 S_t}{D^2 K_t} \right)^{\frac{2\delta}{\sigma^2}} N \left(h_6 \left(\frac{D^2 K_t}{U^2 S_t} \right) \right) \right] \end{aligned}$$

where

$$h_6(z) = \frac{\ln z + (r - \delta + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}.$$

The value of payment $X - K_t$ at time τ_{DUK_t} is equal to $\frac{X - K_t}{K_t} \mathcal{D}\mathcal{S}(S, t; A_8)$ in the case of no-dividends. Calculate this value by dividing into two parts, one with X and the other with $K_t = K_0 e^{\delta t}$ at the barrier hitting time, that is,

$$\begin{aligned} \mathcal{E}(S, t, K_t; A_8) &= X \left(\frac{S_t}{K_t} \right) \left[\left(\frac{U}{D} \right)^{\frac{2r}{\sigma^2} + 1} N \left(-h_6 \left(\frac{U^2 S_t}{D^2 K_t} \right) \right) \right. \\ &\quad \left. + \left(\frac{DK_t}{US_t} \right)^{\frac{2r}{\sigma^2} + 1} \left(\frac{U^2 S_t}{D^2 K_t} \right)^{\frac{2\delta}{\sigma^2}} N \left(h_6 \left(\frac{D^2 K_t}{U^2 S_t} \right) \right) \right] \\ &\quad - K_t \left(\frac{S_t}{K_t} \right) \left[\left(\frac{U}{D} \right)^{\frac{2r}{\sigma^2} + 1} N \left(-h_6 \left(\frac{U^2 S_t}{D^2 K_t} \right) \right) \right. \\ &\quad \left. + \left(\frac{DK_t}{US_t} \right)^{\frac{2r}{\sigma^2} + 1} \left(\frac{U^2 S_t}{D^2 K_t} \right)^{\frac{2\delta}{\sigma^2}} N \left(h_6 \left(\frac{D^2 K_t}{U^2 S_t} \right) \right) \right]. \end{aligned}$$

When the stock pays dividends, as in [Theorem 2.2](#), we adopt V_t to get rid of the dividend term. In order to calculate the first term with constant payment X , we set $V_t = S_t^{\beta - b}$ where b and β are defined in [\(2.2\)](#). For the second term with exponential payment K_t , we note when a payment grows exponentially at rate δ , discounting the payment at the interest rate r is equivalent to discounting a constant payment at the rate $r - \delta$, therefore, set

$$V_t = S_t^{\beta_1 - b_1}$$

where

$$b_1 = \frac{\mu - \delta}{\sigma^2}, \quad \beta_1 = \sqrt{b_1^2 + \frac{2(r - \delta)}{\sigma^2}}.$$

The value of a first-touch digital for the event A_8 is now obtained by following a procedure essentially identical to the proof of [Theorem 2.2](#). \square

4. Conclusion

This paper studies the valuation problem of American barrier option with two barriers. Because a wide variety of traded options are American type, the problem of valuing American options has been an important topic in financial economics. Standard American Options are difficult to price but there exist good numerical or analytical approximation methods. The situation is different for American barrier options. Even, to the best of our knowledge, the literature suggests no approximation formula for American options with two barriers. This paper investigates American barrier options in which two barriers become active alternately.

The analytic valuation formulas for these options are derived by the barrier approximation method under both constant and exponential exercise policies. Our formulas provide a good approximation for the option price in a simple and speedy way.

Appendix A

Let $X_t = \frac{1}{\sigma} \ln(\frac{S_t}{S_0})$ and E^m be the expectation operator under the m -measure. Then X_t is a Brownian motion with drift $\frac{\mu}{\sigma}$. Define τ_d , τ_{du} , and τ_{dul} by stopping times for this process defined as the first time that $X_t = d < X_0$, the first time after τ_d that $X_t = u > d$, and the first time after τ_{du} that $X_t = l < u$ respectively.

Lemma A.1. *For $x \geq l$*

$$\begin{aligned}
 &P(\tau_{dul} \leq T, X_T > x \mid X_0 = 0) \\
 &= \exp\left(\frac{2\mu}{\sigma}(l - u + d)\right) N\left(\frac{2l - 2u + 2d - x + \frac{\mu}{\sigma}T}{\sqrt{T}}\right).
 \end{aligned}$$

Proof. The event $\tau_{dul} \leq T$ is the event that the process X_t falls below d , and then rises above u , and then falls below l before time T . We utilize the reflection principle which reflects the early portion of path prior to its first touch at barriers with respect to the barriers. First, we reflect the original path with respect to d .

The process $X_t = W_t + \frac{\mu}{\sigma}t$ is a standard Brownian motion under the measure Q defined by

$$\frac{dQ}{dP} = \exp\left[-\frac{\mu}{\sigma}W_T - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 T\right].$$

Let us introduce a process $\tilde{X}_t, t \in [0, T]$, defined by the formula

$$\tilde{X}_t = \begin{cases} 2d - X_t & (t \leq \tau_d) \\ X_t & (t > \tau_d). \end{cases}$$

The *reflected path* in Fig. 1 shows this process. Then

$$\begin{aligned}
 &P(\tau_{dul} \leq T, X_T > x \mid X_0 = 0) \\
 &= E^Q \left[\frac{dP}{dQ} \mathbf{1}_{\{\tau_{dul} \leq T, X_T > x\}} \mid X_0 = 0 \right] \\
 &= E^Q \left[e^{\frac{\mu}{\sigma}X_T - \frac{1}{2}\frac{\mu^2}{\sigma^2}T} \mathbf{1}_{\{\tau_{dul} \leq T, X_T > x\}} \mid X_0 = 0 \right] \\
 &= E^Q \left[e^{\frac{\mu}{\sigma}\tilde{X}_T - \frac{1}{2}\frac{\mu^2}{\sigma^2}T} \mathbf{1}_{\{\tau_{ul} \leq T, \tilde{X}_T > x\}} \mid \tilde{X}_0 = 2d \right] \\
 &= \exp\left(\frac{2\mu d}{\sigma}\right) P(\tau_{ul} \leq T, X_T > x \mid X_0 = 2d).
 \end{aligned}$$

We reflect this reflected path before its first touch at u again.

$$\begin{aligned}
 &\exp\left(\frac{2\mu d}{\sigma}\right) P(\tau_{ul} \leq T, X_T > x \mid X_0 = 2d) \\
 &= \exp\left(\frac{2\mu}{\sigma}(u - d)\right) P(\tau_l \leq T, X_T > x \mid X_0 = 2u - 2d).
 \end{aligned}$$

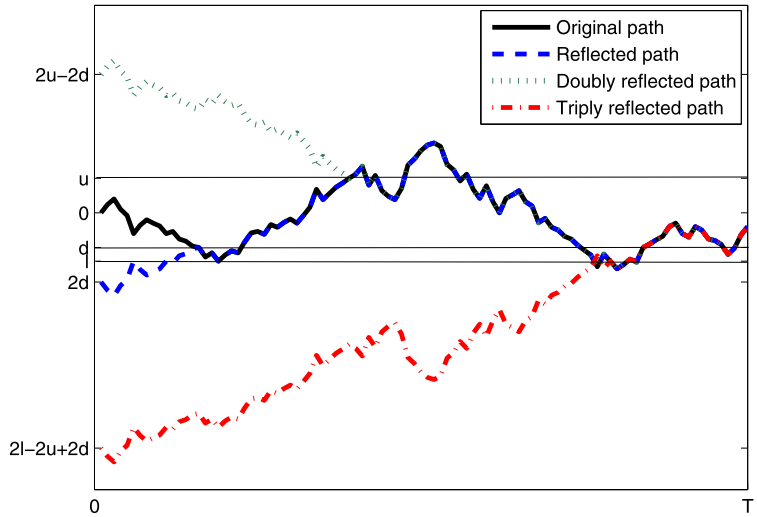


Fig. 1. An illustration of paths using a triple reflection.

We reflect this doubly reflected path before its first touch at l once more. As a result,

$$\begin{aligned} &P(\tau_{dul} \leq T, X_T > x \mid X_0 = 0) \\ &= \exp\left(\frac{2\mu}{\sigma}(l - u + d)\right)P(X_T > x \mid X_0 = 2l - 2u + 2d) \\ &= \exp\left(\frac{2\mu}{\sigma}(l - u + d)\right)N\left(\frac{2l - 2u + 2d - x + \frac{\mu}{\sigma}T}{\sqrt{T}}\right). \quad \square \end{aligned}$$

Lemma A.2. *The probability that the process X_t falls below d , and then rises above u , and then falls below l before time T is*

$$\begin{aligned} P(\tau_{dul} \leq T \mid X_0 = 0) &= \exp\left(\frac{2\mu}{\sigma}(l - u + d)\right)N\left(\frac{l - 2u + 2d + \frac{\mu}{\sigma}T}{\sqrt{T}}\right) \\ &\quad + \exp\left(\frac{2\mu}{\sigma}(u - d)\right)N\left(\frac{l - 2u + 2d - \frac{\mu}{\sigma}T}{\sqrt{T}}\right). \end{aligned}$$

Proof. We note that

$$\begin{aligned} &P(\tau_{dul} \leq T \mid X_0 = 0) \\ &= P(\tau_{dul} \leq T, X_T > l \mid X_0 = 0) + P(\tau_{du} \leq T, X_T \leq l \mid X_0 = 0) \end{aligned}$$

since $\{\tau_{dul} \leq T, X_T \leq l\} = \{\tau_{du} \leq T, X_T \leq l\}$. The first probability of the right-hand side is given by Lemma A.1 with $x = l$ and the second one can be calculated by a similar method to the proof of Lemma A.1. Then we have

$$\begin{aligned} &P(\tau_{dul} \leq T \mid X_0 = 0) \\ &= \exp\left(\frac{2\mu}{\sigma}(d - u + l)\right)N\left(\frac{l - 2u + 2d + \frac{\mu}{\sigma}T}{\sqrt{T}}\right) \\ &\quad + \exp\left(\frac{2\mu d}{\sigma} + \frac{2\mu}{\sigma}(u - 2d)\right)P(X_T \leq l \mid X_0 = 2u - 2d) \end{aligned}$$

$$\begin{aligned}
 &= \exp\left(\frac{2\mu}{\sigma}(d - u + l)\right) N\left(\frac{l - 2u + 2d + \frac{\mu}{\sigma}T}{\sqrt{T}}\right) \\
 &\quad + \exp\left(\frac{2\mu}{\sigma}(u - d)\right) N\left(\frac{l - 2u + 2d - \frac{\mu}{\sigma}T}{\sqrt{T}}\right). \quad \square
 \end{aligned}$$

Lemma A.3. For $x \geq l$, the probability that the process X_t falls below d , and then rises above u , and then does not fall below l before time T , and its value at time T is less than x , is

$$\begin{aligned}
 &P(\tau_{du} < T, \tau_{dul} > T, X_T \leq x \mid X_0 = 0) \\
 &= \exp\left(\frac{2\mu}{\sigma}(u - d)\right) \left[N\left(\frac{x - 2u + 2d - \frac{\mu}{\sigma}T}{\sqrt{T}}\right) - N\left(\frac{l - 2u + 2d - \frac{\mu}{\sigma}T}{\sqrt{T}}\right) \right] \\
 &\quad + \exp\left(\frac{2\mu}{\sigma}(d - u + l)\right) \left[N\left(\frac{2l - 2u + 2d - x + \frac{\mu}{\sigma}T}{\sqrt{T}}\right) - N\left(\frac{l - 2u + 2d + \frac{\mu}{\sigma}T}{\sqrt{T}}\right) \right].
 \end{aligned}$$

Proof.

$$\begin{aligned}
 &P(\tau_{du} < T, \tau_{dul} > T, X_T \leq x \mid X_0 = 0) \\
 &= P(\tau_{du} < T, X_T \leq x \mid X_0 = 0) - P(\tau_{dul} \leq T, X_T \leq x \mid X_0 = 0) \\
 &= P(\tau_{du} < T, X_T \leq x \mid X_0 = 0) - P(\tau_{dul} \leq T \mid X_0 = 0) \\
 &\quad + P(\tau_{dul} \leq T, X_T > x \mid X_0 = 0).
 \end{aligned}$$

Using [Lemma A.1](#) and [Lemma A.2](#), the proof is completed. \square

Theorem A.4. The values of a digital option and a digital share at time t for the event $A_6 = \{t < \tau_D, \tau_{DUk} < T\}$ are

$$\begin{aligned}
 \mathcal{D}(S, t; A_6) &= e^{-r(T-t)} \left[\left(\frac{Dk}{US_t}\right)^{\frac{2\mu}{\sigma^2}} N\left(h_1\left(\frac{D^2k}{U^2S_t}\right)\right) + \left(\frac{U}{D}\right)^{\frac{2\mu}{\sigma^2}} N\left(-h_1\left(\frac{U^2S_t}{D^2k}\right)\right) \right], \\
 \mathcal{DS}(S, t; A_6) &= S_t e^{-q(T-t)} \left[\left(\frac{Dk}{US_t}\right)^{\frac{2\mu}{\sigma^2}} N\left(h_2\left(\frac{D^2k}{U^2S_t}\right)\right) + \left(\frac{U}{D}\right)^{\frac{2\mu}{\sigma^2}} N\left(-h_2\left(\frac{U^2S_t}{D^2k}\right)\right) \right].
 \end{aligned}$$

Proof. Apply [Lemma A.2](#) with $u = \frac{1}{\sigma} \ln \frac{U}{S_t}$, $d = \frac{1}{\sigma} \ln \frac{D}{S_t}$, $l = \frac{1}{\sigma} \ln \frac{k}{S_t}$, and $x = \frac{1}{\sigma} \ln \frac{X}{S_t}$ to derive the risk-neutral probability of early exercise. Then

$$P(t < \tau_D, \tau_{DUk} < T \mid S_t) = \left(\frac{Dk}{US_t}\right)^{\frac{2\mu}{\sigma^2}} N\left(h_1\left(\frac{D^2k}{U^2S_t}\right)\right) + \left(\frac{U}{D}\right)^{\frac{2\mu}{\sigma^2}} N\left(-h_1\left(\frac{U^2S_t}{D^2k}\right)\right).$$

Thus, the value of digital option at time t

$$\mathcal{D}(S, t; A_6) = e^{-r(T-t)} P(t < \tau_D, \tau_{DUk} \leq T)$$

is obtained. Also, the digital share $\mathcal{DS}(S, t; A_6)$ can be valued as in [Theorem 2.1](#). \square

Under a constant exercise policy, the up-and-in chained put option will be exercised early prior to maturity T for $X - k$ if the stock price falls below D , and then rises above U , and then falls to k before time T . Now we consider the value of the first-touch digital at time τ_{DUk} . We examine the case when there is no dividend on the stock first.

Lemma A.5. *If the stock does not pay dividends, the value of a first-touch digital for the event $A_6 = \{t < \tau_D, \tau_{DUk} < T\}$ is*

$$\begin{aligned} \mathcal{E}(S, t, k; A_6) &= \frac{X - k}{k} S_t \left[\left(\frac{Dk}{US_t} \right)^{\frac{2r}{\sigma^2} + 1} N \left(h_3 \left(\frac{D^2 k}{U^2 S_t} \right) \right) + \left(\frac{U}{D} \right)^{\frac{2r}{\sigma^2} + 1} N \left(-h_3 \left(\frac{U^2 S_t}{D^2 k} \right) \right) \right] \end{aligned}$$

where

$$h_3(z) = \frac{\ln z + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}.$$

Proof. The first-touch digital pays $X - k$ at time τ_{DUk} . This money can be used to purchase $\frac{X-k}{k}$ shares of the stock at that time. Since the shares do not pay dividends, it is worth $\frac{X-k}{k} S_T$ at maturity T , i.e.,

$$\mathcal{E}(S, t, k; A_6) = \frac{X - k}{k} \mathcal{DS}(S, t; A_6)$$

where $\mathcal{DS}(S, t; A_6)$ is a value when $q = 0$ in Theorem A.4. \square

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