PRICING CHAINED OPTIONS WITH CURVED BARRIERS

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This paper studies barrier options which are chained together, each with payoff contingent on curved barriers. When the underlying asset price hits a primary curved barrier, a secondary barrier option is given to a primary barrier option holder. Then if the asset price hits another curved barrier, a third barrier option is given, and so on. We provide explicit price formulas for these options when two or more barrier options with exponential barriers are chained together. We then extend the results to the options with general curved barriers.

KEY WORDS: chained option, exponential barrier, curved barrier.

1. INTRODUCTION

Barrier options are popular and widely used path-dependent derivatives because of their flexibility and less expensiveness than vanilla options. Merton (1973) has derived a down-and-out call price by solving the corresponding partial differential equation with some boundary conditions. Rubinstein and Reiner (1991) published closed form pricing formulas for various types of single barrier options. Rich (1994) also provided a mathematical framework to value barrier options. In these papers the underlying asset price is monitored with respect to a single constant barrier for the entire life of the option. For more complicated barrier options, Geman and Yor (1996) followed a probabilistic approach to derive the Laplace transform of the double barrier option price. Heynen and Kat (1994) studied so-called partial barrier options where the underlying price is monitored for a part of the option's lifetime. As a natural variation on the partial barrier structure, window barrier options have become popular with investors, particularly in foreign exchange markets. For a window barrier option, a monitoring period for the barrier commences at the forward starting date and terminates at the early ending date. (We refer to Hui 1997; Guillaume 2003.)

For the exponential barrier whose logarithm is a linear function of time, Kunitomo and Ikeda (1992) derived a pricing formula for double barrier options with curved (exponential) boundaries as the sum of an infinite series. However, for general barriers no closed-form solution is known. Kunitomo and Ikeda (1992) has suggested that it may be possible to approximate a smooth nonlinear function by a set of piecewise exponential functions. Rogers and Zane (1997) has shown that the time-dependent double-barrier option problem for geometric Brownian motion can be reduced to the constant barriers case by transforming the state space and time. A static hedge using calls and puts for a

time-dependent single-barrier option is described in Andersen, Andreasen, and Eliezer (2002). All papers mentioned above are concerned with barrier options where monitoring of the barrier starts at a predetermined date. However, this paper concerns barrier options where monitoring of the other barrier starts at random time when the underlying asset price first crosses a certain exponential barrier level.

The rolling options,[1] which is a variation of barrier option, involve two barriers, both below the initial spot and strike (roll-down) or both above the initial spot and strike (roll-up). For these options, another barrier option with different strike price and barrier of same direction (down or up) is activated when the underlying asset price hits a predetermined barrier (see Gastineau 1994; Carr, Ellis, and Gupta 1998, for details).

This paper concerns options having somewhat similar but different features from rolling options. In other words, in rolling options, another barrier of same direction is activated when the underlying asset price crosses a nearest barrier. However, in this paper, there are two barriers that straddle the initial spot and the other barrier of opposite side is activated at time when the underlying asset crosses one of the two barriers. These options with rather simpler structure were considered in Kwok, Wong, and Lau (2001) and Jun and Ku (2010). Furthermore, this paper studies barrier options where monitoring for the barrier commences at time when the underlying asset price first crosses two exponential barrier levels in a specified order. Since the exponential barriers case includes the constant barriers case (with $\delta_1 = \delta_2 = 0$), this paper can be an extension of Jun and Ku (2010).

In this paper, we derive closed form valuation formulas for various barrier options chained together by applying the reflection principle and Girsanov's Theorem in a proper way when the barrier is given by an exponential function. And we extend the exponential barriers case to the general curved barriers case as suggested in Kunitomo and Ikeda (1992).

Also, the methodology we develop in this paper is easily applicable to a more complicated structure, where more than two hitting times are chained together to activate options with exponential barriers.

The outline of the paper is as follows. Section 2 presents pricing formulas for a down-and-in call option (DIC_u) activated at time when the underlying asset price hits a higher exponential barrier level, and an up-and-in call option (UIC_{ud}) which is activated at time when the asset price crosses two exponential barrier levels (an up-barrier followed by a down-barrier). Section 3 gives the proof of pricing formulas for DIC_u given in Section 2. Finally, Section 4 provides a valuation formula of DIC_u with general curved barriers.

2. CROSSING EXPONENTIAL BARRIERS

Let r be the risk-free interest rate and $\sigma > 0$ be a constant. We assume the price of the underlying asset $S(t)$ follows a geometric Brownian motion

$$S(t) = S(0)\exp((r - \sigma^2/2)t + \sigma W(t),$$

where $W(t)$ is a standard Brownian motion under the risk-neutral probability $\bar{P}$.

Let $X(t) = \frac{1}{\sigma}\ln(S(t)/S(0))$, the upper and lower exponential barriers in the interval [0, T] be $U(t) = Ae^{\delta_1 t}$ and $D(t) = Be^{\delta_2 t}$ ($A > S(0) > B$, $\delta_1 \geq \delta_2$), respectively. We denote by

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[1]A roll-down call is issued with a series of barriers, $H_1 > H_2 > \cdots > H_n$, which are all below the initial spot. At initiation, the roll-down call resembles a European call with strike K_0. If the first barrier H_1 is hit, the strike is rolled down to a new strike $K_1 < K_0$. Upon hitting each subsequent barrier $H_i < H_{i-1}$, the strike is again rolled down to $K_i < K_{i-1}$. When the last barrier H_n is hit, the option knocks out.

E^m the expectation operator under the m-measure and by $\mathcal{F}(t)$ a filtration for Brownian motion $W(t)$.

Consider a European call expiring at T with strike price K. Letting $a = \frac{1}{\sigma}\ln(A/S(0))$ and $b = \frac{1}{\sigma}\ln(B/S(0))$, we define $\bar{\mu} = r/\sigma - \sigma/2$, $\tilde{\mu} = r/\sigma + \sigma/2$, $k = \frac{1}{\sigma}\ln(K/S(0))$,

$$u(t) = \frac{1}{\sigma}\ln\left(\frac{U(t)}{S(0)}\right) = \frac{\delta_1}{\sigma}t + a \quad \text{and} \quad d(t) = \frac{1}{\sigma}\ln\left(\frac{D(t)}{S(0)}\right) = \frac{\delta_2}{\sigma}t + b.$$

We first present the valuation formula for a down-and-in call option commencing at time when the asset price hits the upper exponential barrier $U(t)$. This option gives the option holder the payoff of a call if the price of the underlying asset rises above $U(t)$ and then falls below $D(t)$ before time T, and it pays off zero otherwise. The proof of Theorem 2.1 is given in Section 3.

THEOREM 2.1. *Consider a knock-in call option which is activated at time* $\tau = \min\{t > 0 : S(t) = U(t)\}$. *The value at time* $t_0 < \tau$, DIC_u, *is*

$DIC_u(t_0, S(t_0))$

$$= S(t_0)\left(\frac{B}{A}\right)^{\frac{2}{\sigma^2}(\sigma\tilde{\mu}-\delta_2)}\left(\frac{S(0)}{A}\right)^{\frac{2}{\sigma^2}(\delta_1-\delta_2)}[1_{\{K>D(T)\}}N(z_1) + 1_{\{K\leq D(T)\}}N(z_2)]$$

$$+ S(t_0)\left(\frac{A}{S(0)}\right)^{\frac{2}{\sigma^2}(\sigma\tilde{\mu}-\delta_1)}[1_{\{K\leq D(T)\}}(N(z_3) - N(z_4))]$$

$$- e^{-r(T-t_0)}K\left(\frac{B}{A}\right)^{\frac{2}{\sigma^2}(\sigma\bar{\mu}-\delta_2)}\left(\frac{S(0)}{A}\right)^{\frac{2}{\sigma^2}(\delta_1-\delta_2)}$$

$$\times[1_{\{K>D(T)\}}N(z_1 - \sigma\sqrt{T-t_0}) + 1_{\{K\leq D(T)\}}N(z_2 - \sigma\sqrt{T-t_0})]$$

$$- e^{-r(T-t_0)}K\left(\frac{A}{S(0)}\right)^{\frac{2}{\sigma^2}(\sigma\bar{\mu}-\delta_1)}\left[1_{\{K\leq D(T)\}}\left(N\left(z_3+\sigma\frac{T+t_0}{\sqrt{T-t_0}}\right) - N\left(z_3 + \sigma\frac{T+t_0}{\sqrt{T-t_0}}\right)\right)\right],$$

where

$$z_1 = \frac{1}{\sigma\sqrt{T-t_0}}\ln\left(\frac{B^2 S(t_0)}{A^2 K}\right) + \tilde{\mu}\sqrt{T-t_0} - \frac{2(\delta_1-\delta_2)}{\sigma\sqrt{T-t_0}}t_0,$$

$$z_2 = \frac{1}{\sigma\sqrt{T-t_0}}\ln\left(\frac{B S(t_0)}{A^2}\right) + \left(\tilde{\mu} - \frac{\delta_2}{\sigma}\right)\sqrt{T-t_0} - \frac{2(\delta_1-\delta_2)}{\sigma\sqrt{T-t_0}}t_0,$$

$$z_3 = \frac{1}{\sigma\sqrt{T-t_0}}\ln\left(\frac{B S(t_0)}{A^2}\right) - \frac{1}{\sqrt{T-t_0}}\left(\tilde{\mu}(T+t_0) - \frac{\delta_2}{\sigma}T\right),$$

$$z_4 = \frac{1}{\sigma\sqrt{T-t_0}}\ln\left(\frac{K S(t_0)}{A^2}\right) - \frac{\tilde{\mu}(T+t_0)}{\sqrt{T-t_0}}.$$

$S(0)$ *is the time-zero underlying asset price beyond the lower exponential barrier* $D(0) = B$ *and* $N(x)$ *is the cumulative standard normal distribution function.*

We now consider barrier options activated in the event that the asset price crosses two exponential barriers in a specified order, i.e., the asset price hits the upper exponential barrier $U(t)$ followed by hitting the lower exponential barrier $D(t)$, or vice versa.

The following theorem presents the valuation formula for an up-and-in call option reached by crossing the lower exponential barrier $D(t)$ after crossing the upper exponential barrier $U(t)$. The payoff of this option is a call if the underlying asset price rises above $U(t)$, and then falls below $D(t)$, and then rises above $U(t)$ before time T. Its payoff is zero otherwise.

THEOREM 2.2. *Consider a knock-in call option which is activated at time*

$$\tau_2 = \min\{t > \tau_1 : S(t) = D(t), \ \tau_1 = \min\{t : S(t) = U(t)\}\}.$$

The value at time $t_0 < \tau_1$, UIC_{ud}, is

$UIC_{ud}(t_0, S(t_0))$

$$= S(t_0)\left(\frac{B}{A}\right)^{\frac{2}{\sigma^2}(\sigma\tilde{\mu}-\delta_2)}\left(\frac{S(0)}{A}\right)^{\frac{2}{\sigma^2}(\delta_1-\delta_2)} [1_{\{K\geq U(T)\}}N(z_1) + 1_{\{K<U(T)\}}N(z_5)]$$

$$+ S(t_0)\left(\frac{A^2}{BS(0)}\right)^{\frac{2}{\sigma^2}(\sigma\tilde{\mu}-2\delta_1+\delta_2)} 1_{\{K<U(T)\}}(N(z_6) - N(z_7))$$

$$- e^{-r(T-t_0)}K\left(\frac{B}{A}\right)^{\frac{2}{\sigma^2}(\sigma\bar{\mu}-\delta_2)}\left(\frac{S(0)}{A}\right)^{\frac{2}{\sigma^2}(\delta_1-\delta_2)}$$

$$\times [1_{\{K\geq U(T)\}}N(z_1 - \sigma\sqrt{T-t_0}) + 1_{\{K<U(T)\}}N(z_5 - \sigma\sqrt{T-t_0})]$$

$$- e^{-r(T-t_0)}K\left(\frac{A^2}{BS(0)}\right)^{\frac{2}{\sigma^2}(\sigma\bar{\mu}-2\delta_1+\delta_2)} 1_{\{K<U(T)\}}(N(z_6 + \sigma\sqrt{T-t_0}) - N(z_7 + \sigma\sqrt{T-t_0}))$$

where

$$z_5 = \frac{1}{\sigma\sqrt{T-t_0}}\ln\left(\frac{B^2 S(t_0)}{A^3}\right) + \tilde{\mu}\sqrt{T-t_0} - \frac{\delta_1 T}{\sigma\sqrt{T-t_0}},$$

$$z_6 = \frac{1}{\sigma\sqrt{T-t_0}}\ln\left(\frac{B^2 S(t_0)}{A^3}\right) - \frac{\tilde{\mu}(T+t_0)}{\sqrt{T-t_0}} + \frac{\delta_1 T}{\sigma\sqrt{T-t_0}},$$

$$z_7 = \frac{1}{\sigma\sqrt{T-t_0}}\ln\left(\frac{B^2 K S(t_0)}{A^4}\right) - \frac{\tilde{\mu}(T+t_0)}{\sqrt{T-t_0}}.$$

$S(0)$ is the time-zero underlying asset price beyond the lower exponential barrier $D(0) = B$ at time 0, and $N(x)$ is the cumulative standard normal distribution function.

REMARK 2.3. To value the knock-out call options (DOC_u and UOC_{ud}) which are activated at time

$$\tau = \min\{t : S(t) = U(t), \ U(0) > S(0)\}$$

or time

$$\tau_2 = \min\{t > \tau_1 : S(t) = D(t), \ \tau_1 = \min\{t > 0 : S(t) = U(t)\}, \ U(0) > S(0)\},$$

we apply the general knock-in knock-out parity relation. Thus, we subtract DIC_u from the corresponding up-and-in call price for DOC_u, and subtract UIC_{ud} from the

corresponding DIC_u for UOC_{ud}, i.e.,

$$DOC_u = UIC - DIC_u \text{ and } UOC_{ud} = DIC_u - UIC_{ud}.$$

3. PROOFS

We present in this section the proof of Theorem 2.1. The proof of Theorem 2.2 can be obtained by similar techniques.

Proof of Theorem 2.1. The knock-in call option value at time $t_0 < \tau$ is given by the discounted expected value of its payoff under the risk-neutral measure. Thus

$DIC_u(t_0, S(t_0))$

$$= e^{-r(T-t_0)} E^{\bar{P}}[(S(T) - K)^+ \mathbf{1}_{\{\min_{\tau<t<T}(X(t)-d(t))\le 0,\ \tau\le T,\ S(\tau)=U(\tau)\}}|\mathcal{F}(t_0)]$$

$$= e^{-r(T-t_0)} E^{\bar{P}}[(S(T) - K)\mathbf{1}_{\{\min_{\tau<t<T}(X(t)-d(t))\le 0,\ S(T)>K,\ \tau\le T,\ S(\tau)=U(\tau)\}}|\mathcal{F}(t_0)],$$

where $\mathbf{1}_{\{\}}$ is an indicator function.

Let us define a new measure $\tilde{P}$ such that

$$\frac{d\tilde{P}}{d\bar{P}} = e^{-\frac{1}{2}\sigma^2 T + \sigma W(T)}.$$

Then,

$DIC_u(t_0, S(t_0))$

$$= S(t_0)\tilde{P}(\min_{\tau<t<T}(X(t) - d(t)) \le 0,\ S(T) > K,\ \tau \le T,\ S(\tau) = U(\tau)\,|S(t_0))$$

$$- e^{-r(T-t_0)} K\ \bar{P}(\min_{\tau<t<T}(X(t) - d(t)) \le 0,\ S(T) > K,\ \tau \le T,\ S(\tau) = U(\tau)\,|S(t_0)).$$

It suffices to calculate the required probability under the $\bar{P}$-measure: a simple change of drift from $\bar{\mu}$ to $\tilde{\mu}$ will provide the required probability under the $\tilde{P}$-measure. A process $Y(t) = X(t) - \frac{\delta_1}{\sigma}t = W(t) + (\bar{\mu} - \frac{\delta_1}{\sigma})t$ is a standard Brownian motion under the measure Q, defined by

$$\frac{dQ}{d\bar{P}} = \exp\left(-\left(\bar{\mu} - \frac{\delta_1}{\sigma}\right) W(T) - \frac{1}{2}\left(\bar{\mu} - \frac{\delta_1}{\sigma}\right)^2 T\right).$$

Then

$$\bar{P}\left(\min_{\tau<t<T}(X(t) - d(t)) \le 0,\ X(T) > k,\ \tau \le T,\ X(\tau) = u(\tau)|S(t_0)\right)$$

$$= \bar{P}\left(\min_{\tau<t<T}\left(Y(t) + \frac{\delta_1}{\sigma}t - d(t)\right) \le 0,\ Y(T) > k - \frac{\delta_1}{\sigma}T,\ \tau \le T,\ Y(\tau) = a|S(t_0)\right).$$

Let us introduce a process $\tilde{Y}(t)$, $t \in [0, T]$, defined by the formula

$$\tilde{Y}(t) = \begin{cases} Y(t) & (t \le \tau) \\ 2a - Y(t) & (t > \tau) \end{cases}.$$

By virtue of the reflection principle, the process $\tilde{Y}(t)$ also follows a standard Brownian motion under Q. Then

$$(3.1) \quad \bar{P}\left(\min_{\tau < t < T}\left(Y(t) + \frac{\delta_1}{\sigma}t - d(t)\right) \le 0, \ Y(T) > k - \frac{\delta_1}{\sigma}T, \ \tau \le T, \ Y(\tau) = a \,|\, S(t_0)\right)$$

$$= E^Q\left[e^{\left(\bar{\mu} - \frac{\delta_1}{\sigma}\right)Y(T) - \frac{1}{2}\left(\bar{\mu} - \frac{\delta_1}{\sigma}\right)^2 T}\mathbf{1}_{\left\{\min_{\tau < t < T}\left(Y(t) + \frac{\delta_1}{\sigma}t - d(t)\right) \le 0, \ Y(T) > k - \frac{\delta_1}{\sigma}T, \ \tau \le T, \ Y(\tau) = a\right\}}\,|\,S(t_0)\right]$$

$$= E^Q\left[e^{\left(\bar{\mu} - \frac{\delta_1}{\sigma}\right)(2a - \tilde{Y}(T)) - \frac{1}{2}\left(\bar{\mu} - \frac{\delta_1}{\sigma}\right)^2 T}\mathbf{1}_{\left\{\max_{\tau < t < T}\left(\tilde{Y}(t) - \frac{\delta_1}{\sigma}t + d(t)\right) \ge 2a, \ \tilde{Y}(T) < 2a - k + \frac{\delta_1}{\sigma}T\right\}}\,|\,S(t_0)\right].$$

Another process $Z(t) = \tilde{Y}(t) - \frac{\delta_1 - \delta_2}{\sigma}t$ is a standard Brownian motion under the measure $\tilde{Q}$, defined by

$$\frac{d\tilde{Q}}{dQ} = \exp\left(\left(\frac{\delta_1 - \delta_2}{\sigma}\right)\tilde{Y}(T) - \frac{1}{2}\left(\frac{\delta_1 - \delta_2}{\sigma}\right)^2 T\right).$$

Thus the above equation becomes

$$e^{2a\left(\bar{\mu} - \frac{\delta_1}{\sigma}\right) - \frac{1}{2}\left(\bar{\mu} - \frac{\delta_1}{\sigma}\right)^2 T - \left(\bar{\mu} - \frac{\delta_1}{\sigma}\right)\left(\frac{\delta_1 - \delta_2}{\sigma}\right)T - \frac{1}{2}\left(\frac{\delta_1 - \delta_2}{\sigma}\right)^2 T}$$

$$\times E^{\tilde{Q}}\left[e^{\left(-\bar{\mu} + \frac{\delta_2}{\sigma}\right)Z(T)}\mathbf{1}_{\left\{\max_{0 < t < T}(Z(t)) \ge 2a - b, \ Z(T) < 2a - k + \frac{\delta_2}{\sigma}T\right\}}\,|\,S(t_0)\right].$$

First, we suppose $K > D(T)$, that is, $k > d(T) = \frac{\delta_2}{\sigma}T + b$, and apply the reflection principle again. Let us introduce a process $\tilde{Z}(t)$, $t \in [0, T]$, defined by the formula

$$\tilde{Z}(t) = \begin{cases} Z(t) & (t \le \tau') \\ 2(2a - b) - Z(t) & (t > \tau') \end{cases},$$

where $\tau' = \min\{t > \tau : Z(t) = 2a - b\}$. By virtue of the reflection principle, the process $\tilde{Z}(t)$ also follows a standard Brownian motion under $\tilde{Q}$ and

$$\bar{P}\left(\min_{\tau < t < T}(X(t) - d(t)) \le 0, \ X(T) > k, \ \tau \le T, \ X(\tau) = u(\tau) \,|\, S(t_0)\right)$$

$$= e^{2a\left(\bar{\mu} - \frac{\delta_1}{\sigma}\right) - \frac{1}{2}\left(\bar{\mu} - \frac{\delta_1}{\sigma}\right)^2 T - \left(\bar{\mu} - \frac{\delta_1}{\sigma}\right)\left(\frac{\delta_1 - \delta_2}{\sigma}\right)T - \frac{1}{2}\left(\frac{\delta_1 - \delta_2}{\sigma}\right)^2 T}$$

$$\times E^{\tilde{Q}}\left[e^{\left(-\bar{\mu} + \frac{\delta_2}{\sigma}\right)(4a - 2b - \tilde{Z}(T))}\mathbf{1}_{\left\{\tilde{Z}(T) > 2a - 2b + k - \frac{\delta_2}{\sigma}T\right\}}\,|\,S(t_0)\right]$$

$$= e^{-2a\left(\bar{\mu} + \frac{\delta_1}{\sigma} - \frac{2\delta_2}{\sigma}\right) + 2b\left(\bar{\mu} - \frac{\delta_2}{\sigma}\right)}$$

$$\times E^{\tilde{Q}}\left[e^{\left(\bar{\mu} - \frac{\delta_2}{\sigma}\right)\tilde{Z}(T) - \frac{1}{2}\left(\bar{\mu} - \frac{\delta_2}{\sigma}\right)^2 T}\mathbf{1}_{\left\{\tilde{Z}(T) > 2a - 2b + k - \frac{\delta_2}{\sigma}T\right\}}\,|\,S(t_0)\right].$$

Let us define a probability measure $\widehat{Q}$ by setting

$$\frac{d\widehat{Q}}{d\tilde{Q}} = e^{\left(\bar{\mu}-\frac{\delta_2}{\sigma}\right)\tilde{Z}(T)-\frac{1}{2}\left(\bar{\mu}-\frac{\delta_2}{\sigma}\right)^2 T}$$

so that the process $\widehat{W}(t) = \tilde{Z}(t) - \left(\bar{\mu} - \frac{\delta_2}{\sigma}\right)t$, $t \in [0, T]$, follows a standard Brownian motion under $\widehat{Q}$. Then

(3.2)
$$e^{-2a\left(\bar{\mu}+\frac{\delta_1}{\sigma}-\frac{2\delta_2}{\sigma}\right)+2b\left(\bar{\mu}-\frac{\delta_2}{\sigma}\right)} E^{\tilde{Q}}\left[e^{\left(\bar{\mu}-\frac{\delta_2}{\sigma}\right)\tilde{Z}(T)-\frac{1}{2}\left(\bar{\mu}-\frac{\delta_2}{\sigma}\right)^2 T}\mathbf{1}_{\left\{\tilde{Z}(T)>2a-2b+k-\frac{\delta_2}{\sigma}T\right\}}\,|\,S(t_0)\right]$$

$$= e^{-2a\left(\bar{\mu}+\frac{\delta_1}{\sigma}-\frac{2\delta_2}{\sigma}\right)+2b\left(\bar{\mu}-\frac{\delta_2}{\sigma}\right)}\widehat{Q}\left(\tilde{Z}(T) > 2a - 2b + k - \frac{\delta_2}{\sigma}T\,|\,S(t_0)\right)$$

$$= e^{-2a\left(\bar{\mu}+\frac{\delta_1}{\sigma}-\frac{2\delta_2}{\sigma}\right)+2b\left(\bar{\mu}-\frac{\delta_2}{\sigma}\right)}\widehat{Q}\left(\widehat{W}(T) - \widehat{W}(t_0) > 2a - 2b + k - \bar{\mu}T - \widehat{W}(t_0)\,|\,S(t_0)\right)$$

$$= e^{-2a\left(\bar{\mu}+\frac{\delta_1}{\sigma}-\frac{2\delta_2}{\sigma}\right)+2b\left(\bar{\mu}-\frac{\delta_2}{\sigma}\right)}N\left(\frac{-2a + 2b - k + \bar{\mu}(T-t_0) + \frac{1}{\sigma}\ln\left(\frac{S(t_0)}{S(0)}\right) - \frac{2(\delta_1-\delta_2)}{\sigma}t_0}{\sqrt{T-t_0}}\right)$$

$$= \left(\frac{B}{A}\right)^{\frac{2}{\sigma^2}(\sigma\bar{\mu}-\delta_2)}\left(\frac{S(0)}{A}\right)^{\frac{2}{\sigma^2}(\delta_1-\delta_2)}N\left(\frac{1}{\sigma\sqrt{T-t_0}}\ln\left(\frac{B^2 S(t_0)}{A^2 K}\right)+\bar{\mu}\sqrt{T-t_0}-\frac{2(\delta_1-\delta_2)}{\sigma\sqrt{T-t_0}}t_0\right).$$

In the measure $\tilde{P}$, we follow the same procedures to obtain

$$\tilde{P}\left(\min_{\tau<t<T}(X(t) - d(t)) \le 0,\ S(T) > K,\ \tau \le T,\ S(\tau) = U(\tau)\,|\,S(t_0)\right)$$

$$= \left(\frac{B}{A}\right)^{\frac{2}{\sigma^2}(\sigma\tilde{\mu}-\delta_2)}\left(\frac{S(0)}{A}\right)^{\frac{2}{\sigma^2}(\delta_1-\delta_2)}N\left(\frac{1}{\sigma\sqrt{T-t_0}}\ln\left(\frac{B^2 S(t_0)}{A^2 K}\right)+\tilde{\mu}\sqrt{T-t_0}-\frac{2(\delta_1-\delta_2)}{\sigma\sqrt{T-t_0}}t_0\right).$$

Second, we suppose $K \le D(T)$ in (3.1),

$$\bar{P}\left(\min_{\tau<t<T}\left(Y(t) + \frac{\delta_1}{\sigma}t - d(t)\right) \le 0,\ Y(T) > k - \frac{\delta_1}{\sigma}T,\ \tau \le T,\ Y(\tau) = a\,|\,S(t_0)\right)$$

$$= \bar{P}\left(\min_{\tau<t<T}\left(Y(t) + \frac{\delta_1}{\sigma}t - d(t)\right) \le 0,\ Y(T) > d(T) - \frac{\delta_1}{\sigma}T,\ \tau \le T,\ Y(\tau) = a\,|\,S(t_0)\right)$$

$$+ \bar{P}\left(\min_{\tau<t<T}\left(Y(t) + \frac{\delta_1}{\sigma}t - d(t)\right) \le 0, k - \frac{\delta_1}{\sigma}T < Y(T) \le d(T) - \frac{\delta_1}{\sigma}T,\right.$$

$$\left. \tau \le T,\ Y(\tau) = a\,|\,S(t_0)\right).$$

By the last equation of (3.2),

$$\bar{P}\left(\min_{\tau<t<T}\left(Y(t) + \frac{\delta_1}{\sigma}t - d(t)\right) \le 0,\ Y(T) > d(T) - \frac{\delta_1}{\sigma}T,\ \tau \le T,\ Y(\tau) = a\,|\,S(t_0)\right)$$

$$= \left(\frac{B}{A}\right)^{\frac{2}{\sigma^2}(\sigma\bar{\mu}-\delta_2)}\left(\frac{S(0)}{A}\right)^{\frac{2}{\sigma^2}(\delta_1-\delta_2)}$$

$$\times N\left(\frac{1}{\sigma\sqrt{T-t_0}}\ln\left(\frac{B S(t_0)}{A^2}\right) + \left(\bar{\mu} - \frac{\delta_2}{\sigma}\right)\sqrt{T-t_0} - \frac{2(\delta_1-\delta_2)}{\sigma\sqrt{T-t_0}}t_0\right).$$

When $Y(T) \le d(T) - \frac{\delta_1}{\sigma} T$, the condition $\min_{\tau < t < T} \left(Y(t) + \frac{\delta_1}{\sigma} t - d(t) \right) \le 0$ is always true. The drift of process $Y(t)$ is $\bar{\mu} - \frac{\delta_1}{\sigma}$ with respect to $\bar{P}$ and we use (A.89) in Musiela and Rutkowski (2005, p. 653). Then

$$\bar{P}\left(k - \frac{\delta_1}{\sigma} T < Y(T) \le d(T) - \frac{\delta_1}{\sigma} T, \ \tau \le T, \ Y(\tau) = a \mid S(t_0) \right)$$

$$= e^{2a(\bar{\mu} - \frac{\delta_1}{\sigma})} \left\{ \bar{P}\left(Y(T) \ge 2a - d(T) + \frac{\delta_1}{\sigma} T + 2(\bar{\mu} - \frac{\delta_1}{\sigma})T \mid S(t_0) \right) \right.$$

$$\left. - \bar{P}\left(Y(T) \ge 2a - k + \frac{\delta_1}{\sigma} T + 2(\bar{\mu} - \frac{\delta_1}{\sigma})T \mid S(t_0) \right) \right\}$$

$$= e^{2a(\bar{\mu} - \frac{\delta_1}{\sigma})} \left\{ \bar{P}\left(W(T) - W(t_0) \ge 2a - b + \frac{\delta_1 - \delta_2}{\sigma} T + 2(\bar{\mu} - \frac{\delta_1}{\sigma})T - W(t_0) \mid S(t_0) \right) \right.$$

$$\left. - \bar{P}\left(W(T) - W(t_0) \ge 2a - k + \bar{\mu} T - W(t_0) \mid S(t_0) \right) \right\}$$

$$= \left(\frac{A}{S(0)} \right)^{\frac{2}{\sigma^2}(\sigma\bar{\mu} - \delta_1)} \left\{ N\left(\frac{1}{\sigma\sqrt{T - t_0}} \ln\left(\frac{BS(t_0)}{A^2} \right) - \frac{1}{\sqrt{T - t_0}} \left(\bar{\mu}(T + t_0) - \frac{\delta_2}{\sigma} T \right) \right) \right.$$

$$\left. - N\left(\frac{1}{\sigma\sqrt{T - t_0}} \ln\left(\frac{KS(t_0)}{A^2} \right) - \frac{\bar{\mu}(T + t_0)}{\sqrt{T - t_0}} \right) \right\}.$$

4. CROSSING GENERAL CURVED BARRIERS

We have derived the solutions of valuation problems for the options with exponential barriers in previous sections. In this section, we extend the previous results to the case of general curved barriers by approximating a smooth curve with a set of piecewise exponential functions.

Let us divide the contract interval $[t_0, T]$ into n subintervals $[t_0, t_1)$, $[t_1, t_2)$, ..., $[t_{n-1}, t_n)$, where the length of each interval does not have to be equal and $t_n = T$. For each $i = 0, 1, \ldots, n - 1$, we define

$$A_i(t) = A_i e^{\delta_1(i)(t - t_i)}, \quad t \in [t_i, t_{i+1})$$

with $A_i(t_i) = U(t_i)$ and $A_i(t_{i+1}) = U(t_{i+1})$. Then we approximate a smooth upper curved barrier $U(t)$ by a set of functions $\{A_i(t)\}$. Similarly, we define

$$B_i(t) = B_i e^{\delta_2(i)(t - t_i)}, \quad t \in [t_i, t_{i+1})$$

with $B_i(t_i) = D(t_i)$ and $B_i(t_{i+1}) = D(t_{i+1})$. Then we approximate a smooth lower curved barrier $D(t)$ by a set of functions $\{B_i(t)\}$. We set

$$X(t) = \frac{1}{\sigma} \ln\left(\frac{S(t)}{S(0)} \right), \quad u(t) = \frac{1}{\sigma} \ln\left(\frac{U(t)}{S(0)} \right) \quad \text{and} \quad d(t) = \frac{1}{\sigma} \ln\left(\frac{D(t)}{S(0)} \right).$$

Let

$$u_i(t) = \frac{1}{\sigma} \ln\left(\frac{A_i(t)}{S(0)} \right) = \frac{\delta_1(i)}{\sigma}(t - t_i) + a_i = \frac{a_{i+1} - a_i}{\Delta t_{i+1}}(t - t_i) + a_i$$

where $a_i = \frac{1}{\sigma} \ln\left(\frac{A_i}{S(0)}\right)$, $\Delta t_{i+1} = t_{i+1} - t_i$, and $t \in [t_i, t_{i+1})(i = 0, 1, \ldots, n-1)$. Then $u_i(t)$ is a linear function connecting between $u(t_i) = a_i$ and $u(t_{i+1}) = a_{i+1}$. Also let

$$d_i(t) = \frac{1}{\sigma} \ln\left(\frac{B_i(t)}{S(0)}\right) = \frac{\delta_2(i)}{\sigma}(t - t_i) + b_i = \frac{b_{i+1} - b_i}{\Delta t_{i+1}}(t - t_i) + b_i$$

where $b_i = \frac{1}{\sigma} \ln\left(\frac{B_i}{S(0)}\right)$, $\Delta t_{i+1} = t_{i+1} - t_i$, and $t \in [t_i, t_{i+1})(i = 0, 1, \ldots, n-1)$. Then $d_i(t)$ is a linear function connecting between $d(t_i) = b_i$ and $d(t_{i+1}) = b_{i+1}$. We approximate the upper barrier $u(t)$ and lower barrier $d(t)$ of process $X(t)$ by a set of functions $u_i(t)$ and $d_i(t)$ for $t \in [t_i, t_{i+1})(i = 0, 1, \ldots, n-1)$, respectively.

Denote the slopes of linear function $u_i(t)$ and $d_i(t)$ by

$$m_1(i) = \frac{a_{i+1} - a_i}{\Delta t_{i+1}} \quad \text{and} \quad m_2(i) = \frac{b_{i+1} - b_i}{\Delta t_{i+1}}$$

for the convenience of calculation. We let $\Delta t = \max_i \Delta t_{i+1}$.

In the following, we provide a valuation formula of DIC_u for a down-and-in call option commencing at time when the asset price hits a smooth upper curved barrier. One can derive a valuation formula of UIC_{ud} by the similar method as for DIC_u.

THEOREM 4.1. *Consider a knock-in call option which is activated at time* $\tau = \min\{t : S(t) = U(t)\}$. *The value at time* t_0, DIC_u, *is*

$$DIC_u(t_0, S(t_0)) = S(t_0)\Phi(\tilde{\mu}) - e^{-r(T-t_0)}K\Phi(\bar{\mu}),$$

where

(4.1)
$$\Phi(\mu) = \lim_{\Delta t \to 0} \sum_{0 \le i < j < n} \int_{I_1} \int_{I_2} \cdots \int_{I_{j+1}} \prod_{k=0}^{j} f_k(x_{k+1} | x_k)$$
$$N\left(\frac{-k + x_{j+1} + \mu(T - t_{j+1})}{\sqrt{T - t_{j+1}}}\right) dx_{j+1} \cdots dx_1,$$

$$f_k(x_{k+1}|x_k) = \begin{cases} g_1(x_k, x_{k+1}), & 0 \le k \le i-1 \ (i > 0) \\ g_2(x_k, x_{k+1}), & k = i \\ g_3(x_k, x_{k+1}), & i < k \le j-1 \ (j > i+1) \\ g_4(x_k, x_{k+1}), & k = j \end{cases},$$

$g_1(x_k, x_{k+1})$

$$= \frac{1}{\sqrt{2\pi \Delta t_{k+1}}} \left\{ e^{-\frac{(x_{k+1} - x_k - \mu \Delta t_{k+1})^2}{2\Delta t_{k+1}}} \right.$$
$$\left. - e^{(\mu - m_1(k))(x_{k+1} - x_k - a_{k+1} + a_k) - \frac{1}{2}(\mu - m_1(k))^2 \Delta t_{k+1} - \frac{(a_{k+1} + a_k - x_k - x_{k+1})^2}{2\Delta t_{k+1}}} \right\},$$

$g_2(x_k, x_{k+1})$

$$= \frac{1}{\sqrt{2\pi\,\Delta t_{k+1}}} \left\{ 1_{\{x_{k+1}\geq a_{k+1}\}} e^{-\frac{(x_{k+1}-x_k-\mu\Delta t_{k+1})^2}{2\Delta t_{k+1}}} \right.$$

$$\left. + 1_{\{x_{k+1}<a_{k+1}\}} e^{(\mu-m_1(k))(x_{k+1}-x_k-a_{k+1}+a_k)-\frac{1}{2}(\mu-m_1(k))^2\Delta t_{k+1}-\frac{(a_{k+1}+a_k-x_k-x_{k+1})^2}{2\Delta t_{k+1}}} \right\},$$

$g_3(x_k, x_{k+1})$

$$= \frac{1}{\sqrt{2\pi\,\Delta t_{k+1}}} \left\{ e^{-\frac{(x_{k+1}-x_k-\mu\Delta t_{k+1})^2}{2\Delta t_{k+1}}} \right.$$

$$\left. - e^{(\mu-m_2(k))(x_{k+1}-x_k-b_{k+1}+b_k)-\frac{1}{2}(\mu-m_2(k))^2\Delta t_{k+1}-\frac{(x_{k+1}-x_k-b_{k+1}+b_k)^2}{2\Delta t_{k+1}}} \right\},$$

$g_4(x_k, x_{k+1})$

$$= \frac{1}{\sqrt{2\pi\,\Delta t_{k+1}}} \left\{ 1_{\{x_{k+1}\leq b_{k+1}\}} e^{-\frac{(x_{k+1}-x_k-\mu\Delta t_{k+1})^2}{2\Delta t_{k+1}}} \right.$$

$$\left. + 1_{\{x_{k+1}>b_{k+1}\}} e^{(\mu-m_2(k))(x_{k+1}-x_k-b_{k+1}+b_k)-\frac{1}{2}(\mu-m_2(k))^2\Delta t_{k+1}-\frac{(x_{k+1}-x_k-b_{k+1}+b_k)^2}{2\Delta t_{k+1}}} \right\},$$

and
$$I_k = \begin{cases} (-\infty, a_{k+1}), & 1 \leq k \leq i \ \ (i > 0) \\ (-\infty, \infty), & k = i+1 \\ [\,b_{k+1}, \infty), & i+1 < k \leq j \ \ (j > i+1) \\ (-\infty, \infty), & k = j+1 \end{cases}.$$

Proof. The probability that process $X(t)$ does not cross the linear function $u_k(t)$ in the interval $[t_k, t_{k+1})$ given that the value of $X(t_k)$ is less than a_k is

$$\bar{P}\left(\max_{t_k<t<t_{k+1}} (X(t) - u_k(t)) < 0 | X(t_k) = x_k < a_k, \ X(t_0) = x_0 \right)$$

$$= \bar{P}\left(X(t_{k+1}) < u_k(t_{k+1}) | X(t_k) = x_k < a_k, \ X(t_0) = x_0 \right)$$

$$- \bar{P}\left(\max_{t_k<t<t_{k+1}} (X(t) - u_k(t)) \geq 0, \ X(t_{k+1}) < u_k(t_{k+1}) | X(t_k) = x_k < a_k, \ X(t_0) = x_0 \right)$$

$$= \bar{P}\left(X(t_{k+1}) < a_{k+1} | X(t_i) = x_k < a_k, \ X(t_0) = x_0 \right)$$

$$- \bar{P}\left(\max_{t_k<t<t_{k+1}} (X(t) - m_1(k)(t - t_k)) \geq a_k, \right.$$

$$\left. X(t_{k+1}) - m_1(k)(t_{k+1} - t_k) < a_k | X(t_k) = x_k < a_k, \ X(t_0) = x_0 \right).$$

Then

$$\bar{P}\left(X(t_{k+1}) < a_{k+1} | X(t_k) = x_k < a_k, \ X(t_0) = x_0 \right) = \int_{-\infty}^{a_{k+1}} \frac{1}{\sqrt{2\pi\,\Delta t_{k+1}}} e^{-\frac{(x_{k+1}-x_k-\bar{\mu}\Delta t_{k+1})^2}{2\Delta t_{k+1}}} dx_{k+1}$$

and we set $Y(t) = X(t) - m_1(k)(t - t_k)$ for $t \in [t_k, t_{k+1})$. From Borodin and Salminen (2002, pp. 250–251),

$$\bar{P}\Bigg(\max_{t_k<t<t_{k+1}} (X(t) - m_1(k)(t - t_k)) \geq a_k,$$

$$X(t_{k+1}) - m_1(k)(t_{k+1} - t_k) < a_k \,|\, X(t_k) = x_k < a_k, \ X(t_0) = x_0\Bigg)$$

$$= \bar{P}\Bigg(\max_{t_k<t<t_{k+1}} Y(t) \geq a_k, \ Y(t_{k+1}) < a_k \,|\, Y(t_k) = x_k < a_k, \ X(t_0) = x_0\Bigg)$$

$$= \int_{-\infty}^{a_k} \frac{1}{\sqrt{2\pi\,\Delta t_{k+1}}} e^{(\bar{\mu}-m_1(k))(y-x_k)-\frac{1}{2}(\bar{\mu}-m_1(k))^2\Delta t_{k+1}-\frac{(|y-a_k|+a_k-x_k)^2}{2\Delta t_{k+1}}} \, dy$$

$$= \int_{-\infty}^{a_{k+1}} \frac{1}{\sqrt{2\pi\,\Delta t_{k+1}}} e^{(\bar{\mu}-m_1(k))(x_{k+1}-x_k-a_{k+1}+a_k)-\frac{1}{2}(\bar{\mu}-m_1(k))^2\Delta t_{k+1}-\frac{(a_{k+1}+a_k-x_k-x_{k+1})^2}{2\Delta t_{k+1}}} \, dx_{k+1}$$

where $y = Y(t_{k+1})$ and $x_{k+1} = X(t_{k+1})$. Thus

$$\bar{P}\Bigg(\max_{t_k<t<t_{k+1}} (X(t) - u_k(t)) < 0 \,|\, X(t_k) = x_k < a_k, \ X(t_0) = x_0\Bigg)$$

$$= \int_{-\infty}^{a_{k+1}} \frac{1}{\sqrt{2\pi\,\Delta t_{k+1}}} \Bigg\{ e^{-\frac{(x_{k+1}-x_k-\bar{\mu}\Delta t_{k+1})^2}{2\Delta t_{k+1}}}$$

$$- e^{(\bar{\mu}-m_1(k))(x_{k+1}-x_k-a_{k+1}+a_k)-\frac{1}{2}(\bar{\mu}-m_1(k))^2\Delta t_{k+1}-\frac{(a_{k+1}+a_k-x_k-x_{k+1})^2}{2\Delta t_{k+1}}} \Bigg\} \, dx_{k+1}.$$

Similarly, the probability that process $X(t)$ does not cross the linear function $d_k(t)$ in the interval $[t_k, t_{k+1})$ given that the value of $X(t_k)$ is greater than $b_k = d_k(t_k)$ is

$$\bar{P}\Bigg(\min_{t_k<t<t_{k+1}} (X(t) - d_k(t)) > 0 \,|\, X(t_k) = x_k > b_k, \ X(t_0) = x_0\Bigg)$$

$$= \int_{b_{k+1}}^{\infty} \frac{1}{\sqrt{2\pi\,\Delta t_{k+1}}} \Bigg\{ e^{-\frac{(x_{k+1}-x_k-\bar{\mu}\Delta t_{k+1})^2}{2\Delta t_{k+1}}}$$

$$- e^{(\bar{\mu}-m_2(k))(x_{k+1}-x_k-b_{k+1}+b_k)-\frac{1}{2}(\bar{\mu}-m_2(k))^2\Delta t_{k+1}-\frac{(x_{k+1}-x_k-b_{k+1}+b_k)^2}{2\Delta t_{k+1}}} \Bigg\} \, dx_{k+1}.$$

We then consider the probability that process $X(t)$ crosses the linear function $u_k(t)$ in the interval $[t_k, t_{k+1})$ given $X(t_k) < a_k$. Then

$$\bar{P}\left(\max_{t_k < t < t_{k+1}} (X(t) - u_k(t)) \geq 0 \,|\, X(t_k) = x_k < a_k, \; X(t_0) = x_0\right)$$

$$= \bar{P}\left(X(t_{k+1}) \geq a_{k+1} \,|\, X(t_k) < a_k, \; X(t_0) = x_0\right)$$

$$+ \bar{P}\left(\max_{t_k < t < t_{k+1}} (X(t) - u_k(t)) \geq 0, \; X(t_{k+1}) < a_{k+1} \,|\, X(t_k) < a_k, \; X(t_0) = x_0\right)$$

$$= \int_{a_{k+1}}^{\infty} \frac{1}{\sqrt{2\pi \Delta t_{k+1}}} e^{-\frac{(x_{k+1} - x_k - \bar{\mu}\Delta t_{k+1})^2}{2\Delta t_{k+1}}} \, dx_{k+1}$$

$$+ \int_{-\infty}^{a_{k+1}} \frac{1}{\sqrt{2\pi \Delta t_{k+1}}} e^{(\bar{\mu} - m_1(k))(x_{k+1} - x_k - a_{k+1} + a_k) - \frac{1}{2}(\bar{\mu} - m_1(k))^2 \Delta t_{k+1} - \frac{(a_{k+1} + a_k - x_k - x_{k+1})^2}{2\Delta t_{k+1}}} \, dx_{k+1}.$$

We finally consider the probability that process $X(t)$ crosses the linear function $d_k(t)$ in the interval $[t_k, t_{k+1})$ given $X(t_k) > b_k$.

$$\bar{P}\left(\min_{t_k < t < t_{k+1}} (X(t) - d_k(t)) \leq 0 \,|\, X(t_k) = x_k > b_k, \; X(t_0) = x_0\right)$$

$$= \int_{-\infty}^{b_{k+1}} \frac{1}{\sqrt{2\pi \Delta t_{k+1}}} e^{-\frac{(x_{k+1} - x_k - \bar{\mu}\Delta t_{k+1})^2}{2\Delta t_{k+1}}} \, dx_{k+1}$$

$$+ \int_{b_{k+1}}^{\infty} \frac{1}{\sqrt{2\pi \Delta t_{k+1}}} e^{(\bar{\mu} - m_2(k))(x_{k+1} - x_k - b_{k+1} + b_k) - \frac{1}{2}(\bar{\mu} - m_2(k))^2 \Delta t_{k+1} - \frac{(x_{k+1} - x_k - b_{k+1} + b_k)^2}{2\Delta t_{k+1}}} \, dx_{k+1}.$$

Let i be the smallest k's for which process $X(t)$ crosses the linear function $u_k(t)$ in the interval $[t_k, t_{k+1})$ and j be the smallest k's for which are greater than $i + 1$ and process $X(t)$ crosses the linear function $d_k(t)$ in the interval $[t_k, t_{k+1})$. (If there is no such k to define i or j, then the payoff of the knock-in option is zero, so we don't consider those cases.) Then the joint density function $f(x_1, x_2, \ldots, x_{j+1})$ of $(X(t_1), X(t_2), \ldots, X(t_{j+1}))$ is

$$(4.2) \qquad f(x_1, x_2, \ldots, x_{j+1}) = \prod_{k=0}^{j} f_k(x_{k+1} | x_k) \; (x_0 = X(t_0)),$$

where the conditional density function is written as

$$f_k(x_{k+1} | x_k) = \begin{cases} g_1(x_k, x_{k+1}), & 0 \leq k \leq i - 1 \; (i > 0) \\ g_2(x_k, x_{k+1}), & k = i \\ g_3(x_k, x_{k+1}), & i < k \leq j - 1 \; (j > i + 1) \\ g_4(x_k, x_{k+1}), & k = j \end{cases}.$$

By letting $\tau_u = \min\{t > 0 : X(t) = u(t)\}$ and $\tau_d = \min\{t > \tau_u : X(t) = d(t)\}$,

$$DIC_u(t_0, S(t_0)) = S(t_0) \tilde{P}\left(\min_{\tau_u < t < T} (X(t) - d(t)) \leq 0, \; X(T) > k, \; \tau_u < T \,|\, X(t_0) = x_0\right)$$

$$- e^{-r(T - t_0)} K \, \bar{P}\left(\min_{\tau_u < t < T} (X(t) - d(t)) \leq 0, \; X(T) > k, \; \tau_u < T \,|\, X(t_0) = x_0\right).$$

Considering a simple change of drift, we only need to calculate the probability under measure $\bar{P}$.

$$\bar{P}\left(\min_{\tau_u < t < T}(X(t) - d(t)) \le 0,\ X(T) > k,\ \tau_u < T \mid X(t_0) = x_0\right)$$

$$= \lim_{\Delta t \to 0} \sum_{0 \le i < j < n} \bar{P}\left(\max_{t_k < t < t_{k+1}}(X(t) - u_k(t)) < 0 \text{ for each } k \ (k = 0, 1, \ldots, i-1),\right.$$

$$\tau_u \in [t_i, t_{i+1}),\ \min_{t_k < t < t_{k+1}}(X(t) - d_k(t)) > 0$$

$$\left.\text{for each } k \ (k = i+1, \ldots, j-1),\quad \tau_d \in [t_j, t_{j+1}),\quad X(T) > k \mid X(t_0) = x_0\right)$$

and

(4.3) $\bar{P}\left(X(T) > k \mid X(t_{j+1}) = x_{j+1},\ X(t_0) = x_0\right) = N\left(\dfrac{-k + x_{j+1} + \bar{\mu}(T - t_{j+1})}{\sqrt{T - t_{j+1}}}\right).$

Using equation (4.3) and the joint density function in (4.2),

$$\bar{P}\left(\min_{\tau < t < T}(X(t) - d(t)) \le 0,\ X(T) > k,\ \tau \le T,\ X(\tau) = u(\tau) \mid X(t_0) = x_0\right)$$

$$= \lim_{\Delta t \to 0} \sum_{0 \le i < j < n} \int_{I_1}\int_{I_2}\cdots\int_{I_{j+1}} \prod_{k=0}^{j} f_k(x_{k+1} \mid x_k)$$

$$N\left(\frac{-k + x_{j+1} + \bar{\mu}(T - t_{j+1})}{\sqrt{T - t_{j+1}}}\right) dx_{j+1}\cdots dx_1,$$

where

$$I_k = \begin{cases} (-\infty, a_{k+1}), & 1 \le k \le i \ (i > 0) \\ (-\infty, \infty), & k = i + 1 \\ [\,b_{k+1}, \infty), & i + 1 < k \le j \ (j > i + 1) \\ (-\infty, \infty), & k = j + 1 \end{cases}.$$

$\square$

REMARK 4.2.   The accuracy of approximation (4.1) depends on the size of partition number n. Naturally, the larger n will give the more accurate approximation. For twice differentiable barriers, upper barrier $u(t)$ and lower barrier $d(t)$, Novikov, Frishling, and Kordzakhia (1999) obtained an approximation rate of $O(\sqrt{\log n / n^3})$ by using equally spaced piecewise linear approximation. Furthermore, Pötzelberger and Wang (2001) showed that the approximation error converges to zero at rate $1/n^2$ under an optimal partition of $[t_0, T]$.

REMARK 4.3.   The continuity of $U(t)$ and $D(t)$ have been assumed for the convenience of notation. The result in Theorem 4.1 remains true if one or both of $U(t)$ and $D(t)$ is discontinuous at finite points on $[t_0, T]$ and such that, at any point of discontinuity i, $\lim_{t \to i-} U(t) < \lim_{t \to i+} U(t)$ and $\lim_{t \to i-} D(t) > \lim_{t \to i+} D(t)$. In this case, we only need to include the points of discontinuity in the partition nodes and define values $u(t_i)$ and $d(t_i)$ to be the right-limits of $u(t)$ and $d(t)$ at t_i, respectively.

REFERENCES

ANDERSEN, L., J. ANDREASEN, and D. ELIEZER (2002): Static Replication of Barrier Options: Some General Results, *J. Comput. Finance.* 5, 1–25.

BORODIN, A. N., and P. SALMINEN (2002): *Handbook of Brownian Motion—Facts and Formulae*, 2nd ed., Basel: Birkäuser Verlag.

CARR, P., K. ELLIS, and V. GUPTA (1998): Static Hedging of Exotic Options. *J. Finance* 53(3), 1165–1190.

GASTINEAU, G. (1994): Roll-Up Puts, Roll-Down Calls, and Contingent Premium Options, *J. Derivat.* 1, 40–43.

GEMAN, H., and M. YOR (1996): Pricing and Hedging Double-Barrier Options, Probabilistic Approach, *Math. Finance* 6, 365–378.

GUILLAUME, T. (2003): Window Double Barrier Options, *Rev. Derivat. Res.* 6, 47–75.

HEYNEN, R. C., and H. M. KAT (1994): Partial Barrier Options, *J. Finan. Eng.* 3, 253–274.

HUI, C. H. (1997): Time-Dependent Barrier Option Values, *J. Futures Markets* 17(6), 667–688.

JUN, D., and H. KU (2010): Cross a Barrier to Reach Barrier Options, Working Paper, York University, Toronto, Canada.

KUNITOMO, N., and M. IKEDA (1992): Pricing Options with Curved Boundaries, *Math. Finance* 2, 275–298.

KWOK, Y. K., H. Y. WONG, and K. W. LAU (2001): Pricing Algorithms of Multivariate Path Dependent Options, *J. Complexity* 17, 773–794.

MERTON, R. C. (1973): Theory of Rational Option Pricing, *Bell J. Econ. Manage. Sci.* 4, 141–183.

MUSIELA, M., and M. RUTKOWSKI (2005): *Martingale Methods in Financial Modelling*, 2nd ed., Berlin: Springer.

NOVIKOV, A., V. FRISHLING, and N. KORDZAKHIA (1999): Approximations of Boundary Crossing Probabilities for a Brownian Motion, *J. Appl. Prob.* 99, 1019–1030.

PÖTZELBERGER, K., and L. WANG (2001): Boundary Crossing Probability for Brownian Motion, *J. Appl. Prob.* 38, 152–164.

RICH, D. (1994): The Mathematical Foundations of Barrier Option Pricing Theory, *Adv. Futures Opt. Res.* 7, 267–312.

ROGERS, L. C. G., and O. ZANE (1997): Valuing Moving Barrier Options, *J. Computat. Finance* 1, 5–11.

RUBINSTEIN, M., AND E. REINER (1991): Breaking Down the Barriers, *Risk* 4(8), 28–35.