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# Digital barrier option contract with exponential random time

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In this paper, we derive a closed-form valuation formula for a digital barrier option which is initiated at random time $\tau$ with exponential distribution. We also provide a simple example and graphs to illustrate our result.

*Keywords*: digital barrier option; exponential time.

## 1. Introduction

Barrier options are a widely used class of path-dependent derivative securities. These options either cease to exist or come into existence when some pre-specified asset price barrier is hit during the option's life. Merton (1973) has derived a down-and-out call price by solving the corresponding partial differential equation with some boundary conditions. Reiner & Rubinstein (1991) published closed-form pricing formulas for various types of single barrier options. Rich (1997) also provided a mathematical framework to value barrier options. Moreover, Kunitomo & Ikeda (1992) derived a pricing formula for double barrier options with curved boundaries as the sum of an infinite series. Geman & Yor (1996) followed a probabilistic approach to derive the Laplace transform of the double barrier option price. Pelsser (2000) used contour integration for Laplace transform inversion to price new types of double barrier options. In these papers, the underlying asset price is monitored for barrier hits or crossings during the entire life of the option.

Heynen & Kat (1994) studied partial barrier options where the underlying price is monitored during only part of the option's lifetime. Partial barrier options have two classes. One is forward starting barrier options where the barrier appears at a fixed date strictly after the option's initial starting date. The other is early ending barrier options where the barrier disappears at a specified date strictly before the expiry date. They can be applied for various types of options according to the clients' needs as controlling the starting or ending time of the monitoring period. Also, they can be used as components to synthetically create other types of exotic options.

Digital barrier options (cash-or-nothing barrier options) of knock-in type, a part of standard barrier options, pay out a fixed amount of money if the underlying asset price crosses a given barrier and arrives above a certain value at expiry date. For digital barrier options of knock-out type, the payoff is a fixed amount if the underlying asset price never hits a given barrier and is above a certain level at expiry.

This paper studies a digital barrier option where the monitoring period for barrier hits commences at the exponentially distributed random time, and lasts for the period of length $T$. We note that in the relevant literature, the starting time of monitoring period has been given by a predetermined time. This paper investigates the case that the monitoring period starts at random, which is triggered by an event.

The exponential distribution has been widely used in many areas, including life testing, reliability and operations research due to its nice mathematical form and the characterizing memoryless property. See for example Epstein & Sobel (1954), Marshall & Olkin (1967) and Embrechts & Schmidli (1994). It is well known that an exponential random variable has two important interpretations; one is the waiting time until the first arrival and the other is the lifetime of an item that does not age.

The use of barrier options, digital options and other path-dependent options has increased dramatically by financial institutions. These instruments are used for a wide variety of hedging, risk management and investment purposes. See Goldman *et al.* (1979), Hull (2006, Chapter 22), Rich (1997) and Reiner & Rubinstein (1991), and the references therein for details. Moreover, in recent years, many insurance companies have introduced more innovative designs for their savings products. These products usually have barrier option-like features which allow the policyholder to participate in favourable investment performance while maintaining a floor guarantee on the benefit level.

Suppose that the barrier for the underlying asset price appears by an event, such as default of a bond. We assume the distribution of the time of this event is exponential with mean $1/\lambda$, and its value does not depend on the path of the underlying asset price of the option. In this paper, we derive a closed-form valuation formula for a digital barrier option of knock-out type which initiates at exponential time. Numerical results show the option price as a function of parameter $\lambda$ with different down barriers and strike prices.

This paper is organized as follows. Section 2 presents the explicit price formula for digital knock-out options where monitoring of the barrier starts at time that is exponentially distributed. Section 3 shows an example and graphs to illustrate our result.

## 2. Digital barrier options

Let $r$ be the risk-free interest rate and $\sigma > 0$ be a constant. We assume the price of the underlying asset $S_t$ follows a geometric Brownian motion

$$S_t = S_0 \exp((r - \sigma^2/2)t + \sigma W_t)$$

where $W_t$ is a standard Brownian motion under the risk-neutral probability $P$.

Let $X_t = (1/\sigma)\ln(S_t/S_0)$ and $\mu = r/\sigma - \sigma/2$. Then $X_t = \mu t + W_t$. We define the minimum for $X_t$ to be

$$m_a^b = \inf_{t \in [a,b]} (X_t)$$

and denote by $E^P$ the expectation operator under the $P$-measure and by $\{\mathcal{F}_t\}$ a filtration for Brownian motion $W_t$.

Suppose that $\tau$ is an exponential random variable with parameter $\lambda$ and a barrier option with the lifetime of length $T$ is initiated at time $\tau$, i.e. the monitoring period for asset-price barrier is $[\tau, \tau + T]$. We fix a down barrier $D(< S_0)$, a strike price $K$ and assume $K > D$. We define $d = (1/\sigma)\ln(D/S_0)$ and $k = (1/\sigma)\ln(K/S_0)$.

Consider a digital barrier option of knock-out type which pays out the amount of $A$ when the underlying asset price does not cross the barrier $D$ in $[\tau, \tau + T]$ and is greater than the strike price $K$ at expiration time $\tau + T$. The payoff of this option is zero if the underlying asset price crosses the barrier $D$ or falls below the strike $K$ at expiration. We derive a closed-form formula for the price of this option.

THEOREM 2.1 The value $V$ of a digital knock-out option with the lifetime of length $T$ which initiates at exponential random time $\tau$ with parameter $\lambda$ is

$$\begin{aligned} V = {} & A\,\mathrm{e}^{-rT}\left[\frac{\lambda}{\lambda+r}N(d_1) - f_+^d N\left(d_1 + \frac{d}{\sqrt{T}}\right) - \mathrm{e}^{(\lambda+r)T} f_+^k N(d_2)\right. \\ & \left. + \,\mathrm{e}^{(\lambda+r)T} f_+^k N\left(d_2 + \frac{d}{\sqrt{T}}\right) - \mathrm{e}^{(\lambda+r)T} f_-^k N\left(d_2 + \frac{2k}{\sqrt{T}}\right)\right] \\ & - A\,\mathrm{e}^{-rT}\left(\frac{D}{S_0}\right)^{2\mu/\sigma}\left[\frac{\lambda}{\lambda+r}N\left(d_1 + \frac{2d}{\sqrt{T}}\right) + f_-^{-d} N\left(d_1 + \frac{d}{\sqrt{T}}\right)\right. \\ & - \,\mathrm{e}^{(\lambda+r)T} f_-^{k-2d} N\left(d_2 + \frac{-2d+2k}{\sqrt{T}}\right) + \mathrm{e}^{(\lambda+r)T} f_-^{k-2d} N\left(d_2 + \frac{-d+2k}{\sqrt{T}}\right) \\ & \left. - \,\mathrm{e}^{(\lambda+r)T} f_+^{k-2d} N\left(d_2 + \frac{2d}{\sqrt{T}}\right)\right] \end{aligned}$$

where

$$\begin{aligned} f_\pm^m &= \frac{\lambda}{\sqrt{2(\lambda+r)+\mu^2}\,(\mu \pm \sqrt{2(\lambda+r)+\mu^2})}\,\mathrm{e}^{(\mu \pm \sqrt{2(\lambda+r)+\mu^2})m}, \\ d_1 &= \frac{-k+\mu T}{\sqrt{T}}, \quad d_2 = \frac{-k - T\sqrt{2(\lambda+r)+\mu^2}}{\sqrt{T}} \end{aligned}$$

and $N(\cdot)$ is the cumulative standard normal distribution function.

*Proof.* The digital knock-out option value $V$ at time 0 is given by

$$\begin{aligned} V &= E^P[\mathrm{e}^{-r(\tau+T)} A \mathbf{1}_{\{m_\tau^{\tau+T} > d, S_{\tau+T} > K\}}] \\ &= AE^P[\mathrm{e}^{-r(\tau+T)} \mathbf{1}_{\{m_\tau^{\tau+T} > d, X_{\tau+T} > k\}}] \\ &= A\int_d^\infty \int_0^\infty \lambda\,\mathrm{e}^{-\lambda t}\,\mathrm{e}^{-r(t+T)} P(X_t \in \mathrm{d}z) P(m_t^{t+T} > d,\ X_{t+T} > k|z)\,\mathrm{d}t \\ &= A\int_d^\infty \int_0^\infty \lambda\,\mathrm{e}^{-rT}\,\mathrm{e}^{-(\lambda+r)t}\frac{1}{\sqrt{2\pi t}}\,\mathrm{e}^{-(z-\mu t)^2/2t}\,\mathrm{d}t P(m_0^T > d-z, X_T > k-z)\,\mathrm{d}z. \end{aligned} \tag{1}$$

The inner integral in the last equation of (1) is the form of Laplace transform (see Fusai & Roncoroni, 2008, p. 215). Writing in more detail,

$$\begin{aligned} \int_0^\infty \lambda\,\mathrm{e}^{-rT}\,\mathrm{e}^{-(\lambda+r)t}\frac{1}{\sqrt{2\pi t}}\,\mathrm{e}^{-(z-\mu t)^2/2t}\,\mathrm{d}t &= \frac{\lambda}{\sqrt{2}}\,\mathrm{e}^{\mu z - rT}\int_0^\infty \mathrm{e}^{-(\lambda+r+\mu^2/2)t}\frac{1}{\sqrt{\pi t}}\,\mathrm{e}^{-(\sqrt{2}z)^2/4t}\,\mathrm{d}t \\ &= \frac{\lambda}{\sqrt{2(\lambda+r)+\mu^2}}\,\mathrm{e}^{\mu z - rT - |z|\sqrt{2(\lambda+r)+\mu^2}}. \end{aligned}$$

Thus

$$
\begin{aligned}
V &= A\,\mathrm{e}^{-rT} \int_d^\infty \frac{\lambda}{\sqrt{2(\lambda+r)+\mu^2}}\, \mathrm{e}^{\mu z-|z|\sqrt{2(\lambda+r)+\mu^2}} P(m_0^T > d-z,\ X_T > k-z)\,\mathrm{d}z \\
&= A\,\mathrm{e}^{-rT} \int_d^\infty \frac{\lambda}{\sqrt{2(\lambda+r)+\mu^2}}\, \mathrm{e}^{\mu z-|z|\sqrt{2(\lambda+r)+\mu^2}} \left[ N\left(\frac{-k+z+\mu T}{\sqrt{T}}\right) \right. \\
&\quad \left. -\ \mathrm{e}^{2\mu(d-z)} N\left(\frac{2d-k-z+\mu T}{\sqrt{T}}\right) \right] \mathrm{d}z
\end{aligned}
$$

where the last equation is from the Proposition A.18.3 of Musiela & Rutkowski (2005, p. 656) and $N(\cdot)$ is the cumulative standard normal distribution function.

We have the following equation with two integrals for $V$:

$$
\begin{aligned}
V = A\,\mathrm{e}^{-rT} & \left[ \int_d^\infty \frac{\lambda}{\sqrt{2(\lambda+r)+\mu^2}}\, \mathrm{e}^{\mu z-|z|\sqrt{2(\lambda+r)+\mu^2}} N\left(\frac{-k+z+\mu T}{\sqrt{T}}\right) \mathrm{d}z \right. \\
& \left. - \left(\frac{D}{S_0}\right)^{2\mu/\sigma} \int_d^\infty \frac{\lambda}{\sqrt{2(\lambda+r)+\mu^2}}\, \mathrm{e}^{-\mu z-|z|\sqrt{2(\lambda+r)+\mu^2}} N\left(\frac{2d-k-z+\mu T}{\sqrt{T}}\right) \mathrm{d}z \right].
\end{aligned}
$$

For the first integral

$$
I := \int_d^\infty \frac{\lambda}{\sqrt{2(\lambda+r)+\mu^2}}\, \mathrm{e}^{\mu z-|z|\sqrt{2(\lambda+r)+\mu^2}} N\left(\frac{-k+z+\mu T}{\sqrt{T}}\right) \mathrm{d}z,
$$

since $d < 0$, we can write

$$
\begin{aligned}
I &= \int_d^0 \frac{\lambda}{\sqrt{2(\lambda+r)+\mu^2}}\, \mathrm{e}^{(\mu+\sqrt{2(\lambda+r)+\mu^2})z} N\left(\frac{-k+z+\mu T}{\sqrt{T}}\right) \mathrm{d}z \\
&\quad + \int_0^\infty \frac{\lambda}{\sqrt{2(\lambda+r)+\mu^2}}\, \mathrm{e}^{(\mu-\sqrt{2(\lambda+r)+\mu^2})z} N\left(\frac{-k+z+\mu T}{\sqrt{T}}\right) \mathrm{d}z \\
&= I_1 + I_2.
\end{aligned}
$$

We now calculate $I_1$ and $I_2$ by the integration by parts. Let

$$
a_\pm(\mu) = \frac{\lambda}{\sqrt{2(\lambda+r)+\mu^2}(\mu \pm \sqrt{2(\lambda+r)+\mu^2})}.
$$

Then

$$
\begin{aligned}
I_1 = {} & \left[a_+(\mu)\,\mathrm{e}^{(\mu+\sqrt{2(\lambda+r)+\mu^2})z} N\left(\frac{-k+z+\mu T}{\sqrt{T}}\right)\right]_d^0 \\
& - \int_d^0 a_+(\mu)\,\mathrm{e}^{(\mu+\sqrt{2(\lambda+r)+\mu^2})z} \frac{1}{\sqrt{2\pi T}} \exp\left[-\frac{1}{2}\left(\frac{-k+z+\mu T}{\sqrt{T}}\right)^2\right] \mathrm{d}z \\
= {} & a_+(\mu) N\left(\frac{-k+\mu T}{\sqrt{T}}\right) - a_+(\mu)\,\mathrm{e}^{(\mu+\sqrt{2(\lambda+r)+\mu^2})d} N\left(\frac{-k+d+\mu T}{\sqrt{T}}\right) \\
& - \int_d^0 a_+(\mu) \frac{1}{\sqrt{2\pi T}} \exp\left[(\mu+\sqrt{2(\lambda+r)+\mu^2})z - \frac{1}{2}\left(\frac{-k+z+\mu T}{\sqrt{T}}\right)^2\right] \mathrm{d}z \\
= {} & a_+(\mu) N\left(\frac{-k+\mu T}{\sqrt{T}}\right) - a_+(\mu)\,\mathrm{e}^{(\mu+\sqrt{2(\lambda+r)+\mu^2})d} N\left(\frac{-k+d+\mu T}{\sqrt{T}}\right) \\
& - a_+(\mu)\,\mathrm{e}^{(\lambda+r)T+(\mu+\sqrt{2(\lambda+r)+\mu^2})k} \int_d^0 \frac{1}{\sqrt{2\pi T}} \exp\left[-\frac{1}{2}\left(\frac{z-k-T\sqrt{2(\lambda+r)+\mu^2}}{\sqrt{T}}\right)^2\right] \mathrm{d}z \\
= {} & a_+(\mu) N\left(\frac{-k+\mu T}{\sqrt{T}}\right) - a_+(\mu)\,\mathrm{e}^{(\mu+\sqrt{2(\lambda+r)+\mu^2})d} N\left(\frac{d-k+\mu T}{\sqrt{T}}\right) \\
& - a_+(\mu)\,\mathrm{e}^{(\lambda+r)T+(\mu+\sqrt{2(\lambda+r)+\mu^2})k} N\left(\frac{-k-T\sqrt{2(\lambda+r)+\mu^2}}{\sqrt{T}}\right) \\
& + a_+(\mu)\,\mathrm{e}^{(\lambda+r)T+(\mu+\sqrt{2(\lambda+r)+\mu^2})k} N\left(\frac{d-k-T\sqrt{2(\lambda+r)+\mu^2}}{\sqrt{T}}\right)
\end{aligned}
$$

and

$$
I_2 = -a_-(\mu) N\left(\frac{-k+\mu T}{\sqrt{T}}\right) - a_-(\mu)\,\mathrm{e}^{(\lambda+r)T+(\mu-\sqrt{2(\lambda+r)+\mu^2})k} N\left(\frac{k-T\sqrt{2(\lambda+r)+\mu^2}}{\sqrt{T}}\right).
$$

Since $a_+(\mu) - a_-(\mu) = \lambda/(\lambda+r)$,

$$
\begin{aligned}
I = {} & \frac{\lambda}{\lambda+r} N\left(\frac{-k+\mu T}{\sqrt{T}}\right) - a_+(\mu)\,\mathrm{e}^{(\mu+\sqrt{2(\lambda+r)+\mu^2})d} N\left(\frac{d-k+\mu T}{\sqrt{T}}\right) \\
& - a_+(\mu)\,\mathrm{e}^{(\lambda+r)T+(\mu+\sqrt{2(\lambda+r)+\mu^2})k} N\left(\frac{-k-T\sqrt{2(\lambda+r)+\mu^2}}{\sqrt{T}}\right) \\
& + a_+(\mu)\,\mathrm{e}^{(\lambda+r)T+(\mu+\sqrt{2(\lambda+r)+\mu^2})k} N\left(\frac{d-k-T\sqrt{2(\lambda+r)+\mu^2}}{\sqrt{T}}\right) \\
& - a_-(\mu)\,\mathrm{e}^{(\lambda+r)T+(\mu-\sqrt{2(\lambda+r)+\mu^2})k} N\left(\frac{k-T\sqrt{2(\lambda+r)+\mu^2}}{\sqrt{T}}\right).
\end{aligned}
$$

Similarly, we compute the second integral for $V$

$$\begin{aligned}
&\int_d^{\infty} \frac{\lambda}{\sqrt{2(\lambda+r)+\mu^2}}\, \mathrm{e}^{-\mu z-|z|\sqrt{2(\lambda+r)+\mu^2}} N\left(\frac{2d-k-z+\mu T}{\sqrt{T}}\right) \mathrm{d}z \\
&= \frac{\lambda}{\lambda+r} N\left(\frac{2d-k+\mu T}{\sqrt{T}}\right) + a_-(\mu)\, \mathrm{e}^{(\mu-\sqrt{2(\lambda+r)+\mu^2})(-d)} N\left(\frac{d-k+\mu T}{\sqrt{T}}\right) \\
&\quad - a_-(\mu)\, \mathrm{e}^{(\lambda+r)T+(\mu-\sqrt{2(\lambda+r)+\mu^2})(k-2d)} N\left(\frac{-2d+k-T\sqrt{2(\lambda+r)+\mu^2}}{\sqrt{T}}\right) \\
&\quad + a_-(\mu)\, \mathrm{e}^{(\lambda+r)T+(\mu-\sqrt{2(\lambda+r)+\mu^2})(k-2d)} N\left(\frac{-d+k-T\sqrt{2(\lambda+r)+\mu^2}}{\sqrt{T}}\right) \\
&\quad - a_+(\mu)\, \mathrm{e}^{(\lambda+r)T+(\mu+\sqrt{2(\lambda+r)+\mu^2})(k-2d)} N\left(\frac{2d-k-T\sqrt{2(\lambda+r)+\mu^2}}{\sqrt{T}}\right).
\end{aligned}$$

□

REMARK 2.2 Consider a digital knock-out barrier option which is initiated at the occurrence of an event. Suppose that the event has not yet occurred at time $t_0 > 0$. Then the distribution of its payoff function (conditioned on $\mathcal{F}_{t_0}$) is the same as the one at time 0, which is implied by the memoryless property of exponential distribution and the Markov property of $S_t$. Thus the value $V$ in Theorem 2.1 remains the same, except $S_{t_0}$ in replacement of $S_0$, at time $t_0$.

REMARK 2.3 In this section, we treat a down-and-out digital option. The methodology we develop also works well for the prices of up-and-in, up-and-out and down-and-in options.

## 3. Example and graphs

### 3.1 *Example*

In this subsection, we present an example to explain how Theorem 2.1 is applied to value a financial contract.

Suppose Companies A and B make a deal: Company B makes a payment of $\$10^6$ to Company A in the event of Company A's default. This contract contains clauses requiring that the stock price of Company B does not fall below the barrier level $D$ for the period of length $T$ starting from the time of Company A's default, and at the end of that period, the stock price is above the strike level $K$. This condition is interpreted as Company B is not in financial difficulties for a certain period in the event of Company A's default.

Assume that default time of Company A has the exponential distribution with parameter $\lambda$. We note that $\lambda$ might be determined by historical default data according to Company A's bond rating. It might also be given through the negotiations between the companies. We assume that the stock price of Company B follows a geometric Brownian motion with initial price $S_0 = 100$ and volatility $= 0.3$. Also, in this example, $D = 80$, $K = 100$, $T = 0.5$, $r = 0$ and $\lambda = 0.001$. Then how much is the premium that Company A should pay to Company B for this contract?

By the price formula in Theorem 2.1, we have the premium $V = \$20{,}042$ to the nearest dollar.

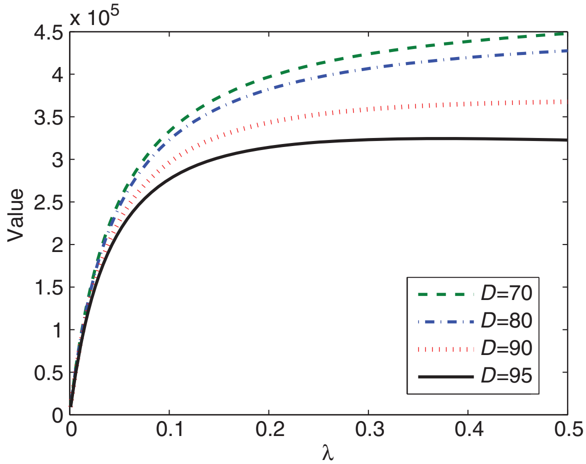


FIG. 1. Digital barrier option value with varying $\lambda$ and down barrier $D$ (option parameters: $S_0 = 100, K = 100, A = 10^6$, $\sigma = 0.3, r = 0.05$ and $T = 0.5$).

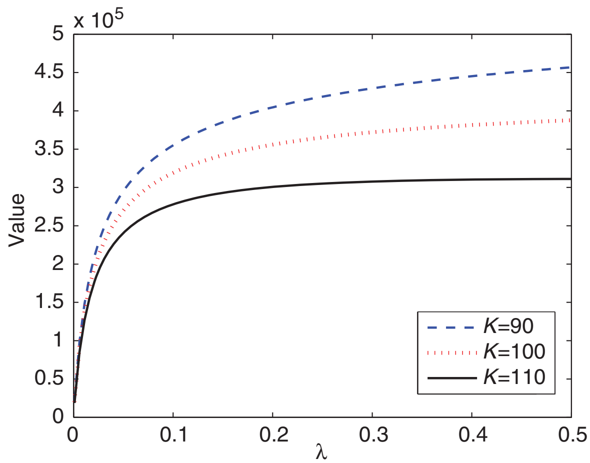


FIG. 2. Digital barrier option value with varying $\lambda$ and strike price $K$ (option parameters: $S_0 = 100, D = 80, A = 10^6, \sigma = 0.3$, $r = 0.05$ and $T = 0.5$).

### 3.2 *Graphs*

We illustrate the properties of our solution obtained in Theorem 2.1. Figure 1 shows how the digital knock-out option value $V$ changes when the parameter $\lambda$ varies from 0.001 to 0.5 and down barrier $D$ takes the values of 70, 80 90 or 95. The other parameters are given by $S_0 = 100,\ K = 100,\ A = 10^6,\ \sigma = 0.3,\ r = 0.05$ and $T = 0.5$. Figure 2 shows how the digital barrier option value $V$ changes when the parameter $\lambda$ varies from 0.001 to 0.5 and strike price $K$ takes the values of 90, 100 or 110. The other parameters are given by $S_0 = 100,\ D = 80,\ A = 10^6,\ \sigma = 0.3,\ r = 0.05$ and $T = 0.5$.

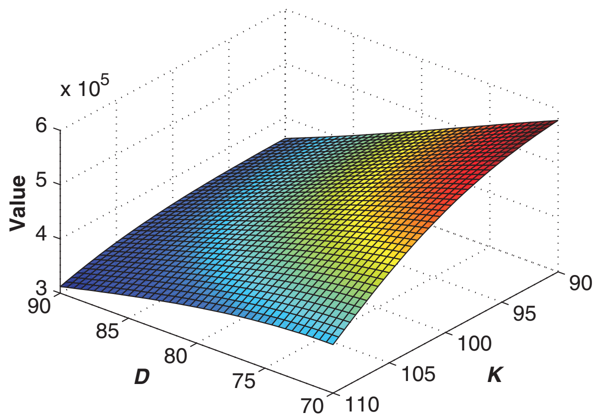


FIG. 3. Digital barrier option values with varying down barrier $D$ and strike price $K$ (option parameters: $S_0 = 100, \lambda = 1, A = 10^6, \sigma = 0.3, r = 0.05$ and $T = 0.5$).

In Figure 1, we observe that the digital knock-out option value $V$ increases as $\lambda$ increases for $D = 70, 80$ or 90. An interpretation of this is as follows: if the exponential parameter $\lambda$ increases, $\tau$ is more likely to be small. Then, the probability of the underlying asset price crossing the down barrier $D$ in the time period of length $T$ gets smaller, which results in the increase of $V$. However, if the down barrier gets closer to initial stock price $S_0$, this argument does not work and $V$ may not increase (see the graph for $D = 95$).

In Fig. 2, we observe if the strike price $K$ takes the larger value, the digital knock-out option price becomes smaller. This property of $V$ is the same as that of the regular barrier option price.

Also, one can expect for a fixed $\lambda$, $V$ increases when $D$ or $K$ decreases. Figure 3 demonstrates this desiring property of $V$.

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