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Cross a barrier to reach barrier options

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ABSTRACT

This paper studies a new type of barrier options where a regular barrier option comes into existence in the event that the underlying asset price first crosses specified barrier levels. We derive closed form formulas for the prices via the reflection principle and provide numerical results to illustrate the properties of our solutions with respect to option parameters.

1. Introduction

Barrier options are a widely used class of path-dependent derivative securities. These options "knock in" or "knock out" when the price of the underlying asset crosses a certain barrier level. For example, an up-and-in call option gives the option holder the payoff of a call if the price of the underlying asset reaches a higher barrier level during the option's life, and it pays off zero unless the asset price reaches that level. For an up-and-out call, the option becomes worthless if the underlying asset price hits a higher barrier, and its payoff at expiration is a call otherwise. Options with a lower barrier level are said to be down-and-in and down-and-out options.

Merton [6] has derived a down-and-out call price by solving the corresponding partial differential equation with some boundary conditions. Rubinstein and Reiner [10] published closed form pricing formulas for various types of single barrier options. Rich [9] also provided a mathematical framework to value barrier options. In these papers, the underlying asset price is monitored with respect to a single constant barrier for the entire life of the option.

Due to their popularity in a market, more complicated structures of barrier options have been studied by a number of authors. Kunitomo and Ikeda [5] derived a pricing formula for double barrier options with curved boundaries as the sum of an infinite series. Geman and Yor [1] followed a probabilistic approach to derive the Laplace transform of the double barrier option price.

Heynan and Kat [3] studied so-called partial barrier options where the underlying price is monitored for a part of the option's lifetime. For theses options, either the barrier disappears at a specified date strictly before the maturity (i.e., early ending option) or the barrier appears at a fixed date strictly after the start of the option (i.e., forward starting option). In the paper, the authors gave valuation formulas for partial barrier options in terms of bivariate normal distribution functions. As a natural variation on the partial barrier structure, window barrier options have become popular wit h investors, particularly in foreign exchange markets. For a window barrier option, a monitoring period for the barrier commences at the forward start date and terminates at the early ending date. (We refer to Hui [4] and Guillaume [2].) However, all these papers are concerned with barrier options where monitoring of the barrier starts at a predetermined date.

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This paper concerns barrier options where monitoring of the barrier starts at random time when the underlying asset price first crosses a certain barrier level. For these options, one can consider as a secondary barrier option that is given to a primary barrier option holder in the event of the primary barrier being crossed. Furthermore, this paper studies barrier options where monitoring for the barrier commences at time when the underlying asset price first crosses two barrier levels in a specified order. Interestingly, these options can be seen as three standard barrier options that are chained together. Options with having the similar features to the former (simple) case was discussed in Pfeffer [8], where the price is computed by Laplace transforms through conditioning on the hitting time. However, the technique adopted in [8] cannot be applied well to the latter case of this paper, in which two hitting times are involved to activate a barrier option.

In this paper, we derive closed form valuation formulas for various barrier options of this new type by applying the reflection principle and Girsanov's Theorem in a proper way. Also, the methodology we develop in this paper is easily applicable to more complicated structure, where more than two hitting times are chained together to activate barrier options.

This paper is organized as follows. Section 2 presents a valuation formula for a down-and-in call option (DIC_u) activated at time when the underlying asset price hits a higher barrier level. The prices of the options with knock-out barrier are discussed. Section 3 gives a valuation formula of an up-and-in call option (UIC_{ud}) which is activated at time when the asset price crosses two barrier levels (an up-barrier followed by a down-barrier). The case of knock-out options is also treated. Section 4 shows the numerical results of six graphs explaining the properties of the prices DIC_u and UIC_{ud} with respect to option parameters. The pricing formulas for an up-and-in call option reached by crossing a down-barrier (UIC_d) and a down-and-in call option reached by crossing a down-barrier followed by an up-barrier (DIC_{du}) are given in Appendices A and B.

2. Case of crossing a barrier

Let r be the risk-free interest rate and $\sigma > 0$ be a constant. We assume the price of the underlying asset S follows a geometric Brownian motion

$$S_t = S_0 \exp(\overline{\mu} t + \sigma W_t)$$

where $\overline{\mu} = r - \frac{\sigma^2}{2}$ and W_t is a standard Brownian motion under the risk-neutral probability $\overline{P}$.

Let $X_t = \frac{1}{\sigma}\ln(S_t/S_0)$. We define the minimum and maximum for X_t to be

$$m_a^b = \inf_{t\in[a,b]}(X_t), \qquad M_a^b = \sup_{t\in[a,b]}(X_t)$$

and denote by E^m the expectation operator under the m-measure.

Consider a European call expiring at T with strike price K. We fix an up-barrier U ($> S_0$) and a down-barrier D ($< S_0$). We define $k = \frac{1}{\sigma}\ln(K/S_0)$, $u = \frac{1}{\sigma}\ln(U/S_0)$ and $d = \frac{1}{\sigma}\ln(D/S_0)$.

Now we provide the valuation formula for a down-and-in call option commencing at time when the asset price hits the up-barrier U under the assumption of $K > D$; For the case of $K \leqslant D$, see Theorem 2.2.

Theorem 2.1. *Suppose $K > D$. The knock-in call option value at time 0, DIC_u, which is activated at time $\tau = \min\{t\colon S_t = U,\ U > S_0\}$ is*

$$DIC_u = S_0\left(\frac{D}{U}\right)^{\frac{2\widetilde{\mu}}{\sigma^2}} N(z_1) - e^{-rT}K\left(\frac{D}{U}\right)^{\frac{2\overline{\mu}}{\sigma^2}} N(z_1 - \sigma\sqrt{T})$$

where

$$z_1 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2 S_0}{U^2 K}\right) + \frac{\widetilde{\mu}}{\sigma}\sqrt{T},$$

$\widetilde{\mu} = r + \frac{\sigma^2}{2}$, *$S_0$ is the underlying asset spot value at time 0 beyond the down-barrier D and $N(x)$ is the cumulative standard normal distribution function.*

Proof. The knock-in call option value at time 0 is given by the discounted expected value of its payoff under the risk-neutral measure. Thus

$$DIC_u = e^{-rT}E^{\overline{P}}\left[(S_T - K)^+ \mathbf{1}_{\{m_\tau^T \leqslant d,\, \tau \leqslant T\, S_\tau = U\}}\right] = e^{-rT}E^{\overline{P}}\left[(S_T - K)\mathbf{1}_{\{m_\tau^T \leqslant d, S_T > K,\, \tau \leqslant T\, S_\tau = U\}}\right]$$

where $\mathbf{1}_{\{\}}$ is an indicator function.

Let us define a new measure $\widetilde{P}$ such that

$$\frac{d\widetilde{P}}{d\overline{P}} = e^{-\frac{1}{2}\sigma^2 T + \sigma W_T}.$$

Then,

$$DIC_u = S_0 \widetilde{P}\big(m_\tau^T \leqslant d,\ S_T > K,\ \tau \leqslant T\ S_\tau = U\big) - e^{-rT} K \overline{P}\big(m_\tau^T \leqslant d,\ S_T > K,\ \tau \leqslant T\ S_\tau = U\big).$$

It suffices to calculate the required probability under the $\overline{P}$-measure: a simple change of drift from $\overline{\mu}$ to $\widetilde{\mu}$ will provide the required probability under the $\widetilde{P}$-measure. Note that

$$\overline{P}\big(m_\tau^T \leqslant d,\ S_T > K,\ \tau \leqslant T\ S_\tau = U\big) = \overline{P}\big(m_\tau^T \leqslant d,\ X_T > k,\ \tau \leqslant T,\ X_\tau = u\big)$$

where $X_t = W_t + (\frac{\overline{\mu}}{\sigma})t$ is a standard Brownian motion under the equivalent measure Q, defined by

$$\frac{dQ}{d\overline{P}} = \exp\left[-\frac{\overline{\mu}}{\sigma} W_T - \frac{1}{2}\left(\frac{\overline{\mu}}{\sigma}\right)^2 T\right].$$

Let us introduce a process $\widetilde{X}_t$, $t \in [0, T]$, defined by the formula

$$\widetilde{X}_t = \begin{cases} X_t & (t \leqslant \tau) \\ 2u - X_t & (t > \tau). \end{cases}$$

By virtue of the reflection principle, the process $\widetilde{X}_t$ also follows a standard Brownian motion under Q. Then

$$\overline{P}\big(m_\tau^T \leqslant d,\ X_T > k,\ \tau \leqslant T,\ X_\tau = u\big) \tag{1}$$
$$= E^{\overline{P}}[\mathbf{1}_{\{m_\tau^T \leqslant d,\, X_T > k,\, \tau \leqslant T,\, X_\tau = u\}}] \tag{2}$$
$$= E^Q\left[\frac{d\overline{P}}{dQ}\mathbf{1}_{\{m_\tau^T \leqslant d,\, X_T > k,\, \tau \leqslant T,\, X_\tau = u\}}\right] \tag{3}$$
$$= E^Q\big[e^{\frac{\overline{\mu}}{\sigma}X_T - \frac{1}{2}\frac{\overline{\mu}^2}{\sigma^2}T}\mathbf{1}_{\{m_\tau^T \leqslant d,\, X_T > k,\, \tau \leqslant T,\, X_\tau = u\}}\big] \tag{4}$$
$$= E^Q\big[e^{\frac{\overline{\mu}}{\sigma}(2u - \widetilde{X}_T) - \frac{1}{2}\frac{\overline{\mu}^2}{\sigma^2}T}\mathbf{1}_{\{\widetilde{M}_\tau^T \geqslant 2u-d,\, \widetilde{X}_T < 2u-k,\, \tau \leqslant T\}}\big] \tag{5}$$

where $\widetilde{M}_\tau^T = \sup_{t\in[\tau,T]}(\widetilde{X}_t)$.

Since $2u - d > u$, $\{\widetilde{M}_\tau^T \geqslant 2u - d,\ \tau \leqslant T\} = \{\widetilde{M}_0^T \geqslant 2u - d\}$. Thus,

$$\overline{P}\big(m_\tau^T \leqslant d,\ X_T > k,\ \tau \leqslant T,\ X_\tau = u\big) = E^Q\big[e^{\frac{\overline{\mu}}{\sigma}(2u - \widetilde{X}_T) - \frac{1}{2}\frac{\overline{\mu}^2}{\sigma^2}T}\mathbf{1}_{\{\widetilde{M}_0^T \geqslant 2u-d,\, \widetilde{X}_T < 2u-k\}}\big].$$

We apply the reflection principle again. Let us introduce a process $\widehat{X}_t$, $t \in [0, T]$, defined by the formula

$$\widehat{X}_t = \begin{cases} \widetilde{X}_t & (t \leqslant \tau') \\ 2(2u - d) - \widetilde{X}_t & (t > \tau') \end{cases}$$

where $\tau' = \min\{t > \tau\colon\ \widetilde{X}_t = 2u - d\}$. By virtue of the reflection principle, the process $\widehat{X}_t$ also follows a standard Brownian motion under Q and

$$\begin{aligned}
&\overline{P}\big(m_\tau^T \leqslant d,\ X_T > k,\ \tau \leqslant T,\ X_\tau = u\big) \\
&\quad = E^Q\big[e^{\frac{\overline{\mu}}{\sigma}(-2u+2d+\widehat{X}_T) - \frac{1}{2}\frac{\overline{\mu}^2}{\sigma^2}T}\mathbf{1}_{\{\widehat{X}_T > 2u-2d+k\}}\big] \\
&\quad = e^{\frac{\overline{\mu}}{\sigma}(-2u+2d)}E^Q\big[e^{\frac{\overline{\mu}}{\sigma}\widehat{X}_T - \frac{1}{2}\frac{\overline{\mu}^2}{\sigma^2}T}\mathbf{1}_{\{\widehat{X}_T > 2u-2d+k\}}\big].
\end{aligned}$$

Let us define an equivalent probability measure $\widetilde{Q}$ by setting

$$\frac{d\widetilde{Q}}{dQ} = e^{\frac{\overline{\mu}}{\sigma}\widehat{X}_T - \frac{1}{2}\frac{\overline{\mu}^2}{\sigma^2}T}$$

so that the process $\widetilde{W}_t = \widehat{X}_t - \frac{\overline{\mu}}{\sigma}t$, $t \in [0, T]$, follows a standard Brownian motion under $\widetilde{Q}$.

$$e^{\frac{\overline{\mu}}{\sigma}(-2u+2d)}E^Q\big[e^{\frac{\overline{\mu}}{\sigma}\widehat{X}_T - \frac{1}{2}\frac{\overline{\mu}^2}{\sigma^2}T}\mathbf{1}_{\{\widehat{X}_T > 2u-2d+k\}}\big] \tag{6}$$
$$= e^{\frac{\overline{\mu}}{\sigma}(-2u+2d)}\widetilde{Q}\,(\widehat{X}_T > 2u - 2d + k) \tag{7}$$
$$= e^{\frac{\overline{\mu}}{\sigma}(-2u+2d)}\widetilde{Q}\left(\widetilde{W}_T > 2u - 2d + k - \frac{\overline{\mu}}{\sigma}T\right) \tag{8}$$

$$= e^{\frac{\overline{\mu}}{\sigma}2(-u+d)} N\left(\frac{-2u+2d-k+\frac{\overline{\mu}}{\sigma}T}{\sqrt{T}}\right) \tag{9}$$

$$= \left(\frac{D}{U}\right)^{\frac{2\overline{\mu}}{\sigma^2}} N\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2 S_0}{U^2 K}\right)+\frac{\overline{\mu}}{\sigma}\sqrt{T}\right). \tag{10}$$

In the measure $\widetilde{P}$, we follow the same process to obtain

$$\widetilde{P}\left(m_\tau^T \leqslant d,\ X_T > k,\ \tau \leqslant T,\ X_\tau = u\right) = \left(\frac{D}{U}\right)^{\frac{2\widetilde{\mu}}{\sigma^2}} N\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2 S_0}{U^2 K}\right)+\frac{\widetilde{\mu}}{\sigma}\sqrt{T}\right).$$

Therefore

$$DIC_u = S_0\left(\frac{D}{U}\right)^{\frac{2\widetilde{\mu}}{\sigma^2}} N\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2 S_0}{U^2 K}\right)+\frac{\widetilde{\mu}}{\sigma}\sqrt{T}\right) - e^{-rT}K\left(\frac{D}{U}\right)^{\frac{2\overline{\mu}}{\sigma^2}} N\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2 S_0}{U^2 K}\right)+\frac{\overline{\mu}}{\sigma}\sqrt{T}\right). \qquad \square$$

Theorem 2.2. *Suppose $K \leqslant D$. The knock-in call option value at time 0, DIC_u, which is activated at time $\tau = \min\{t:\ S_t = U,\ U > S_0\}$ is*

$$DIC_u = S_0\left[\left(\frac{D}{U}\right)^{\frac{2\widetilde{\mu}}{\sigma^2}} N(z_2) + \left(\frac{U}{S_0}\right)^{\frac{2\widetilde{\mu}}{\sigma^2}} N(z_3) - \left(\frac{U}{S_0}\right)^{\frac{2\widetilde{\mu}}{\sigma^2}} N(z_4)\right]$$
$$- e^{-rT}K\left[\left(\frac{D}{U}\right)^{\frac{2\overline{\mu}}{\sigma^2}} N(z_2-\sigma\sqrt{T}) + \left(\frac{U}{S_0}\right)^{\frac{2\overline{\mu}}{\sigma^2}} N(z_3+\sigma\sqrt{T}) - \left(\frac{U}{S_0}\right)^{\frac{2\overline{\mu}}{\sigma^2}} N(z_4+\sigma\sqrt{T})\right]$$

where

$$z_2 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{DS_0}{U^2}\right)+\frac{\widetilde{\mu}}{\sigma}\sqrt{T}, \qquad z_3 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{DS_0}{U^2}\right)-\frac{\widetilde{\mu}}{\sigma}\sqrt{T},$$
$$z_4 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{KS_0}{U^2}\right)-\frac{\widetilde{\mu}}{\sigma}\sqrt{T}.$$

Proof. We start from (1) in the proof of Theorem 2.1.

$$\overline{P}\left(m_\tau^T \leqslant d,\ X_T > k,\ \tau \leqslant T,\ X_\tau = u\right) = \overline{P}\left(m_\tau^T \leqslant d,\ X_T > d,\ \tau \leqslant T,\ X_\tau = u\right) + \overline{P}\left(m_\tau^T \leqslant d,\ k < X_T \leqslant d,\ \tau \leqslant T,\ X_\tau = u\right).$$

By Eq. (10),

$$\overline{P}\left(m_\tau^T \leqslant d,\ X_T > d,\ \tau \leqslant T,\ X_\tau = u\right) = \left(\frac{D}{U}\right)^{\frac{2\overline{\mu}}{\sigma^2}} N\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{DS_0}{U^2}\right)+\frac{\overline{\mu}}{\sigma}\sqrt{T}\right).$$

From (A.89) in Musiela and Rutkowski [7, p. 653], we obtain

$$\overline{P}\left(m_\tau^T \leqslant d,\ k < X_T \leqslant d,\ \tau \leqslant T,\ X_\tau = u\right)$$
$$= \overline{P}(X_\tau = u,\ \tau \leqslant T,\ k < X_T \leqslant d)$$
$$= e^{\frac{2\overline{\mu}}{\sigma}u}\left\{\overline{P}\left(X_T \geqslant 2u-d+2\frac{\overline{\mu}}{\sigma}T\right) - \overline{P}\left(X_T \geqslant 2u-k+2\frac{\overline{\mu}}{\sigma}T\right)\right\}$$
$$= \left(\frac{U}{S_0}\right)^{\frac{2\overline{\mu}}{\sigma^2}}\left\{N\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{DS_0}{U^2}\right)-\frac{\overline{\mu}}{\sigma}\sqrt{T}\right) - N\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{KS_0}{U^2}\right)-\frac{\overline{\mu}}{\sigma}\sqrt{T}\right)\right\}. \qquad \square$$

Remark 2.3. To value the knock-out call (down-and-out) option at time 0, DOC_u, which is activated at time $\tau = \min\{t:\ S_t = U, U > S_0\}$, we apply the knock-in knock-out parity relation. So, we subtract DIC_u from the up-and-in call price UIC to get

$$DOC_u = UIC - DIC_u.$$

The valuation formulas for an up-and-in call (UIC_d) and an up-and-out call (UOC_d) activated in the event that the asset price first hits the down-barrier D are provided in Appendix A.

3. Case of crossing two barrier levels

We derived, in the previous section, the pricing formulas for barrier options commencing at time when the asset price crosses a specified barrier level. In this section, we consider barrier options activated in the event that the asset price crosses two barrier levels in a specified order, i.e., hits the up-barrier U followed by reaching the down-barrier D, or vice versa.

The following theorem presents the valuation formula for an up-and-in call option reached by crossing the down-barrier after crossing the up-barrier under the assumption of $K < U$. Theorem 3.2 gives the formula for the case of $K \geqslant U$.

Theorem 3.1. *Suppose $K < U$. The knock-in call option value at time 0, UIC_{ud}, which is activated at time*

$$\tau_2 = \min\{t > \tau_1\colon\ S_t = D,\ \tau_1 = \min\{t > 0\colon\ S_t = U,\ U > S_0\}\}$$

is

$$UIC_{ud} = S_0\left[\left(\frac{D}{U}\right)^{\frac{2\widetilde{\mu}}{\sigma^2}} N(z_5) + \left(\frac{U^2}{DS_0}\right)^{\frac{2\widetilde{\mu}}{\sigma^2}} N(z_6) - \left(\frac{U^2}{DS_0}\right)^{\frac{2\widetilde{\mu}}{\sigma^2}} N(z_7)\right] - e^{-rT}K\left[\left(\frac{D}{U}\right)^{\frac{2\overline{\mu}}{\sigma^2}} N(z_5 - \sigma\sqrt{T}) + \left(\frac{U^2}{DS_0}\right)^{\frac{2\overline{\mu}}{\sigma^2}} N(z_6 + \sigma\sqrt{T}) - \left(\frac{U^2}{DS_0}\right)^{\frac{2\overline{\mu}}{\sigma^2}} N(z_7 + \sigma\sqrt{T})\right]$$

where

$$z_5 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2S_0}{U^3}\right) + \frac{\widetilde{\mu}}{\sigma}\sqrt{T}, \qquad z_6 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2S_0}{U^3}\right) - \frac{\widetilde{\mu}}{\sigma}\sqrt{T},$$
$$z_7 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2KS_0}{U^4}\right) - \frac{\widetilde{\mu}}{\sigma}\sqrt{T}$$

S_0 is the underlying asset spot value at time 0 beyond the down-barrier D, and $N(x)$ is the cumulative standard normal distribution function.

Proof. Under the risk-neutral measure, the knock-in call option value at time 0 is

$$UIC_{ud} = e^{-rT}E^{\overline{P}}\big[(S_T - K)^+\mathbf{1}_{\{M_{\tau_2}^T \geqslant u,\ \tau_1<\tau_2\leqslant T,\ S_{\tau_1}=U,\ S_{\tau_2}=D\}}\big]$$
$$= e^{-rT}E^{\overline{P}}\big[(S_T - K)\mathbf{1}_{\{M_{\tau_2}^T \geqslant u,\ S_T>K,\ \tau_1<\tau_2\leqslant T,\ S_{\tau_1}=U,\ S_{\tau_2}=D\}}\big].$$

Let us define a new measure $\widetilde{P}$ such that

$$\frac{d\widetilde{P}}{d\overline{P}} = e^{-\frac{1}{2}\sigma^2T+\sigma W_T}.$$

Then, we have

$$UIC_{ud} = S_0\widetilde{P}\big(M_{\tau_2}^T \geqslant u,\ S_T > K,\ \tau_1 < \tau_2 \leqslant T,\ S_{\tau_1} = U,\ S_{\tau_2} = D\big) - e^{-rT}K\overline{P}\big(M_{\tau_2}^T \geqslant u,\ S_T > K,\ \tau_1 < \tau_2 \leqslant T,\ S_{\tau_1} = U,\ S_{\tau_2} = D\big).$$

We calculate the required probability only under the $\overline{P}$-measure as in the proof of Theorem 2.1. Note that

$$\overline{P}\big(M_{\tau_2}^T \geqslant u,\ S_T > K,\ \tau_1 < \tau_2 \leqslant T,\ S_{\tau_1} = U,\ S_{\tau_2} = D\big)$$
$$= \overline{P}\big(M_{\tau_2}^T \geqslant u,\ X_T > k,\ \tau_1 < \tau_2 \leqslant T,\ X_{\tau_1} = u,\ X_{\tau_2} = d\big)$$

where $X_t = W_t + (\frac{\overline{\mu}}{\sigma})t$ is a standard Brownian motion under the equivalent probability measure Q, defined by

$$\frac{dQ}{d\overline{P}} = \exp\left[-\frac{\overline{\mu}}{\sigma}W_T - \frac{1}{2}\left(\frac{\overline{\mu}}{\sigma}\right)^2T\right].$$

Then,

$$\overline{P}\big(M_{\tau_2}^T \geqslant u,\ X_T > k,\ \tau_1 < \tau_2 \leqslant T,\ X_{\tau_1} = u,\ X_{\tau_2} = d\big)$$
$$= E^Q\left[\frac{d\overline{P}}{dQ}\mathbf{1}_{\{M_{\tau_2}^T \geqslant u,\ X_T>k,\ \tau_1<\tau_2\leqslant T,\ X_{\tau_1}=u,\ X_{\tau_2}=d\}}\right]$$
$$= E^Q\big[e^{\frac{\overline{\mu}}{\sigma}X_T - \frac{1}{2}\frac{\overline{\mu}^2}{\sigma^2}T}\mathbf{1}_{\{M_{\tau_2}^T \geqslant u,\ X_T>k,\ \tau_1<\tau_2\leqslant T,\ X_{\tau_1}=u,\ X_{\tau_2}=d\}}\big].$$

Let us introduce a process $\widetilde{X}_t$, $t \in [0, T]$, defined by the formula

$$\widetilde{X}_t = \begin{cases} X_t & (t \leqslant \tau_1) \\ 2u - X_t & (t > \tau_1). \end{cases}$$

By virtue of the reflection principle, the process $\widetilde{X}_t$ also follows a standard Brownian motion under Q, and

$$E^Q\left[e^{\frac{\overline{\mu}}{\sigma}X_T - \frac{1}{2}\frac{\overline{\mu}^2}{\sigma^2}T}\mathbf{1}_{\{M_{\tau_2}^T \geqslant u,\, X_T > k,\, \tau_1 < \tau_2 \leqslant T,\, X_{\tau_1} = u,\, X_{\tau_2} = d\}}\right]$$
$$= E^Q\left[e^{\frac{\overline{\mu}}{\sigma}(2u - \widetilde{X}_T) - \frac{1}{2}\frac{\overline{\mu}^2}{\sigma^2}T}\mathbf{1}_{\{\widetilde{m}_{\tau_2}^T \leqslant u,\, \widetilde{X}_T < 2u-k,\, \tau_2 \leqslant T,\, \widetilde{X}_{\tau_2} = 2u-d\}}\right]$$

where $\widetilde{m}_{\tau_2}^T = \inf_{t \in [\tau_2, T]}(\widetilde{X}_t)$.

Here, we apply the reflection principle again. Let us introduce a process $\widehat{X}_t$, $t \in [0, T]$, defined by the formula

$$\widehat{X}_t = \begin{cases} \widetilde{X}_t & (t \leqslant \tau_2) \\ 2(2u - d) - \widetilde{X}_t & (t > \tau_2). \end{cases}$$

Then, the process $\widehat{X}_t$ also follows a standard Brownian motion under Q, and

$$E^Q\left[e^{\frac{\overline{\mu}}{\sigma}(2u - \widetilde{X}_T) - \frac{1}{2}\frac{\overline{\mu}^2}{\sigma^2}T}\mathbf{1}_{\{\widetilde{m}_{\tau_2}^T \leqslant u,\, \widetilde{X}_T < 2u-k,\, \tau_2 \leqslant T,\, \widetilde{X}_{\tau_2} = 2u-d\}}\right]$$
$$= E^Q\left[e^{\frac{\overline{\mu}}{\sigma}(-2u + 2d + \widehat{X}_T) - \frac{1}{2}\frac{\overline{\mu}^2}{\sigma^2}T}\mathbf{1}_{\{\widehat{M}_{\tau_2}^T \geqslant 3u-2d,\, \widehat{X}_T > 2u-2d+k,\, \tau_2 \leqslant T\}}\right]$$

where $\widehat{M}_{\tau_2}^T = \sup_{t \in [\tau_2, T]}(\widehat{X}_t)$.

Since $3u - 2d > 2u - d > u$, $\{\widehat{M}_{\tau_2}^T \geqslant 3u - 2d,\ \tau_2 \leqslant T\} = \{\widehat{M}_0^T \geqslant 3u - 2d\}$. Thus,

$$E^Q\left[e^{\frac{\overline{\mu}}{\sigma}(-2u + 2d + \widehat{X}_T) - \frac{1}{2}\frac{\overline{\mu}^2}{\sigma^2}T}\mathbf{1}_{\{\widehat{M}_{\tau_2}^T \geqslant 3u-2d,\, \widehat{X}_T > 2u-2d+k,\, \tau_2 \leqslant T\}}\right]$$
$$= e^{\frac{\overline{\mu}}{\sigma}(-2u+2d)}E^Q\left[e^{\frac{\overline{\mu}}{\sigma}\widehat{X}_T - \frac{1}{2}\frac{\overline{\mu}^2}{\sigma^2}T}\mathbf{1}_{\{\widehat{M}_0^T \geqslant 3u-2d,\, \widehat{X}_T > 2u-2d+k\}}\right].$$

Let us define an equivalent probability measure $\widetilde{Q}$ by setting

$$\frac{d\widetilde{Q}}{dQ} = e^{\frac{\overline{\mu}}{\sigma}\widehat{X}_T - \frac{1}{2}\frac{\overline{\mu}^2}{\sigma^2}T}$$

so that the process $\widetilde{W}_t = \widehat{X}_t - \frac{\overline{\mu}}{\sigma}t,\ t \in [0, T]$, follows a standard Brownian motion under $\widetilde{Q}$.

$$e^{\frac{\overline{\mu}}{\sigma}(-2u+2d)}E^Q\left[e^{\frac{\overline{\mu}}{\sigma}\widehat{X}_T - \frac{1}{2}\frac{\overline{\mu}^2}{\sigma^2}T}\mathbf{1}_{\{\widehat{M}_0^T \geqslant 3u-2d,\, \widehat{X}_T > 2u-2d+k\}}\right]$$
$$= e^{\frac{\overline{\mu}}{\sigma}(-2u+2d)}\widetilde{Q}\left(\widehat{M}_0^T \geqslant 3u - 2d,\ \widehat{X}_T > 2u - 2d + k\right)$$
$$= e^{\frac{\overline{\mu}}{\sigma}(-2u+2d)}\left[\widetilde{Q}\left(\widehat{M}_0^T \geqslant 3u - 2d\right) - \widetilde{Q}\left(\widehat{M}_0^T \geqslant 3u - 2d,\ \widehat{X}_T \leqslant 2u - 2d + k\right)\right].$$

From (A.92) and (A.90) in Musiela and Rutkowski [7, p. 655], we obtain

$$\widetilde{Q}\left(\widehat{M}_0^T \geqslant 3u - 2d\right) = \widetilde{Q}\left(\widehat{X}_T \geqslant 3u - 2d\right) + e^{\frac{2\overline{\mu}}{\sigma}(3u-2d)}\widetilde{Q}\left(\widehat{X}_T \geqslant 3u - 2d + \frac{2\overline{\mu}}{\sigma}T\right)$$
$$= \widetilde{Q}\left(\widetilde{W}_T \geqslant 3u - 2d - \frac{\overline{\mu}}{\sigma}T\right) + e^{\frac{2\overline{\mu}}{\sigma}(3u-2d)}\widetilde{Q}\left(\widetilde{W}_T \geqslant 3u - 2d + \frac{\overline{\mu}}{\sigma}T\right)$$
$$= N\left(\frac{-3u + 2d}{\sqrt{T}} + \frac{\overline{\mu}}{\sigma}\sqrt{T}\right) + e^{\frac{2\overline{\mu}}{\sigma}(3u-2d)}N\left(\frac{-3u + 2d}{\sqrt{T}} - \frac{\overline{\mu}}{\sigma}\sqrt{T}\right)$$
$$= N\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2 S_0}{U^3}\right) + \frac{\overline{\mu}}{\sigma}\sqrt{T}\right) + e^{\frac{2\overline{\mu}}{\sigma}(3u-2d)}N\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2 S_0}{U^3}\right) - \frac{\overline{\mu}}{\sigma}\sqrt{T}\right)$$

and

$$\widetilde{Q}\left(\widehat{M}_0^T \geqslant 3u - 2d,\ \widehat{X}_T \leqslant 2u - 2d + k\right) = e^{\frac{2\overline{\mu}}{\sigma}(3u-2d)}N\left(\frac{-4u + 2d + k}{\sqrt{T}} - \frac{\overline{\mu}}{\sigma}\sqrt{T}\right)$$
$$= e^{\frac{2\overline{\mu}}{\sigma}(3u-2d)}N\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2 K S_0}{U^4}\right) - \frac{\overline{\mu}}{\sigma}\sqrt{T}\right).$$

Therefore,

$$\begin{aligned}
&\overline{P}\big(M_{\tau_2}^T \geqslant u,\ X_T > k,\ \tau_1 < \tau_2 \leqslant T,\ X_{\tau_1} = u,\ X_{\tau_2} = d\big) \\
&= \left(\frac{D}{U}\right)^{\frac{2\overline{\mu}}{\sigma^2}} N\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2 S_0}{U^3}\right) + \frac{\overline{\mu}}{\sigma}\sqrt{T}\right) + \left(\frac{U^2}{DS_0}\right)^{\frac{2\overline{\mu}}{\sigma^2}} N\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2 S_0}{U^3}\right) - \frac{\overline{\mu}}{\sigma}\sqrt{T}\right) \\
&\quad - \left(\frac{U^2}{DS_0}\right)^{\frac{2\overline{\mu}}{\sigma^2}} N\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2 K S_0}{U^4}\right) - \frac{\overline{\mu}}{\sigma}\sqrt{T}\right).
\end{aligned}$$

In the measure $\widetilde{P}$, we get the similar results.

$$\begin{aligned}
&\widetilde{P}\big(M_{\tau_2}^T \geqslant u,\ X_T > k,\ \tau_1 < \tau_2 \leqslant T,\ X_{\tau_1} = u,\ X_{\tau_2} = d\big) \\
&= \left(\frac{D}{U}\right)^{\frac{2\widetilde{\mu}}{\sigma^2}} N\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2 S_0}{U^3}\right) + \frac{\widetilde{\mu}}{\sigma}\sqrt{T}\right) + \left(\frac{U^2}{DS_0}\right)^{\frac{2\widetilde{\mu}}{\sigma^2}} N\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2 S_0}{U^3}\right) - \frac{\widetilde{\mu}}{\sigma}\sqrt{T}\right) \\
&\quad - \left(\frac{U^2}{DS_0}\right)^{\frac{2\widetilde{\mu}}{\sigma^2}} N\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2 K S_0}{U^4}\right) - \frac{\widetilde{\mu}}{\sigma}\sqrt{T}\right).
\end{aligned}$$

By combining the results together, we complete the proof. □

If the strike price K is greater than or equal to the up-barrier U, the payoff of the call is zero unless the asset price at expiry is greater than the up-barrier U. Since the asset price must reach the up-barrier after crossing the down-barrier before expiry for nonzero payoff, UIC_{ud} is equal to DIC_u in this case.

Theorem 3.2. *Suppose $K \geqslant U$. The knock-in call option value at time 0, UIC_{ud}, which is activated at time*

$$\tau_2 = \min\big\{t > \tau_1:\ S_t = D,\ \tau_1 = \min\{t > 0:\ S_t = U,\ U > S_0\}\big\}$$

is

$$UIC_{ud} = S_0\left(\frac{D}{U}\right)^{\frac{2\widetilde{\mu}}{\sigma^2}} N(z_1) - e^{-rT} K\left(\frac{D}{U}\right)^{\frac{2\overline{\mu}}{\sigma^2}} N(z_1 - \sigma\sqrt{T}).$$

Remark 3.3. The knock-out call option value at time 0, UOC_{ud}, which is activated at time $\tau_2 = \min\{t > \tau_1:\ S_t = D,\ \tau_1 = \min\{t > 0:\ S_t = U,\ U > S_0\}\}$ is calculated as follows:

$$UOC_{ud} = DIC_u - UIC_{ud}.$$

The valuation formulas for a down-and-in call (DIC_{du}) and a down-and-out call (DOC_{du}) activated in the event of crossing the down-barrier followed by crossing the up-barrier are provided in Appendix B.

4. Numerical results

In this section, we illustrate the properties of our solutions obtained in Sections 2 and 3. Fig. 1 shows how the price DIC_u changes when the volatility and strike price vary. The volatility increases from 0.1 to 0.5 and the strike price decreases from 120 to 80 under the assumption that $S_0 = 100$, $U = 110$, $D = 90$, $r = 0.05$ and $T = 0.5$. We see that the option price DIC_u increases as the volatility increases or the strike price decreases. Since the probability of the asset price hitting given barriers gets bigger as the volatility increases, the knock-in option value naturally increases.

Fig. 2 concerns the price UIC_{ud}, and shows the same properties as DIC_u in Fig. 1. Fig. 3 shows the comparison of DIC_u and UIC_{ud} with the same parameters as in Figs. 1 and 2. We observe UIC_{ud} is always lower than DIC_u as expected.

Fig. 4 illustrates the changes of UIC_{ud} when the up-barrier and down-barrier vary. With the parameters $S_0 = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.3$, $T = 0.5$, U changes between 100 and 120, and D moves between 80 and 100. We observe from Fig. 4, UIC_{ud} approximates to zero as both U and D drifts farther from the underlying asset price at time 0. If U and D are set to be $S_0 = 100$, then UIC_{ud} is equal to the vanilla call option price, which corresponds to the highest point in Fig. 4. Fig. 5 shows the prices DIC_u and UIC_{ud} for different underlying asset prices S_0.

Finally, Fig. 6 represents that simulation results, up to 10,000 paths of the underlying asset price process, converge to the exact value obtained in Section 3. We simulate the monitoring frequency from 1000 to 10,000 under the assumption that $S_0 = 100$, $K = 100$, $U = 110$, $D = 90$, $r = 0.05$, $\sigma = 0.3$, $T = 0.5$. Then the exact value from Theorem 3.1 is 0.2146 and the simulation value of UIC_{ud} is 0.2142 with monitoring frequency of 10,000. In general, a standard Monte Carlo Method systematically underprice the knock-in call option (see Geman and Yor [1]). The reason is that the underlying asset price

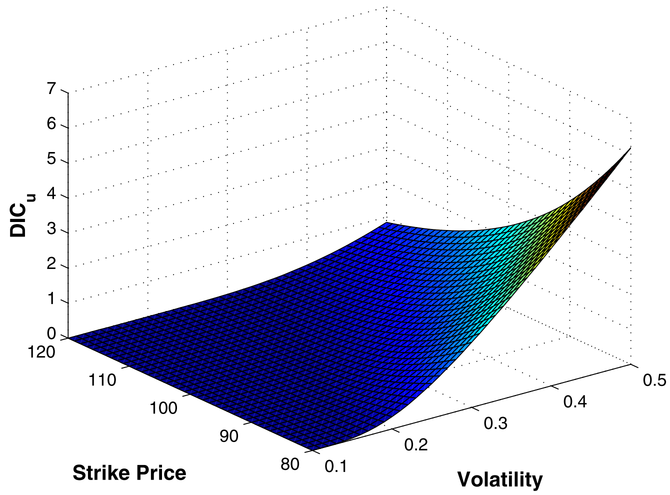

Fig. 1. DIC_u result, varying K and σ (option parameters: $S_0 = 100$, $U = 110$, $D = 90$, $r = 0.05$, and $T = 0.5$).

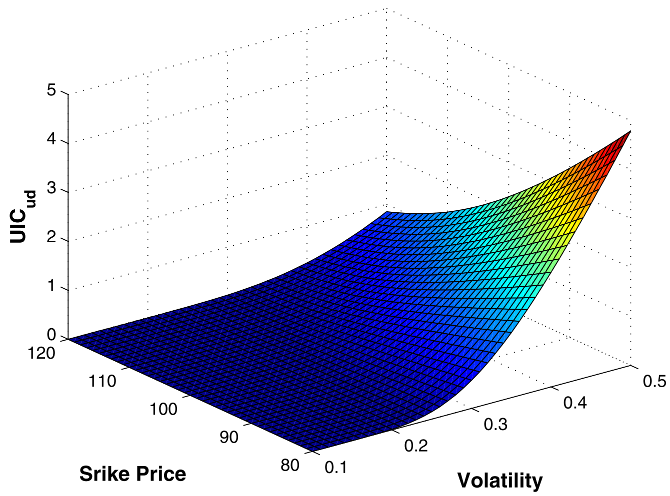

Fig. 2. UIC_{ud} result, varying K and σ (option parameters: $S_0 = 100$, $U = 110$, $D = 90$, $r = 0.05$, and $T = 0.5$).

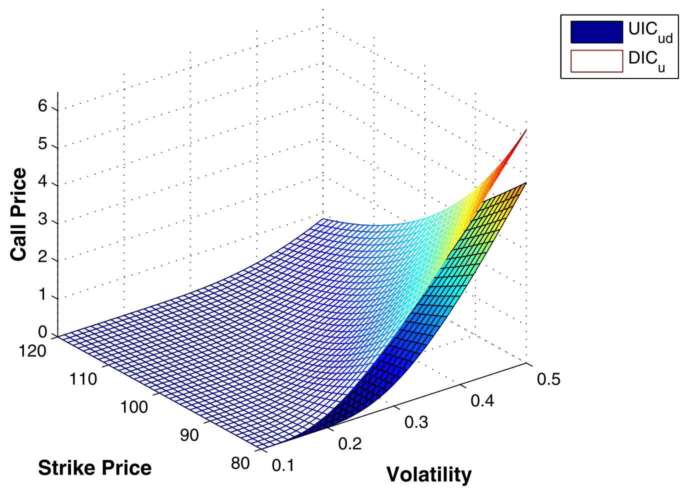

Fig. 3. Comparison between UIC_{ud} and DIC_u, varying K and σ (option parameters: $S_0 = 100$, $U = 110$, $D = 90$, $r = 0.05$, and $T = 0.5$).

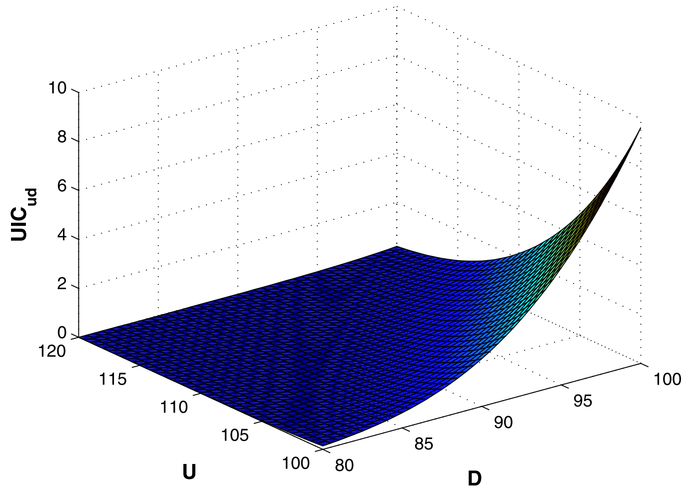

Fig. 4. UIC_{ud} results when U varies between 100 and 120, and D varies between 80 and 100 (option parameters: $S_0 = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.3$, and $T = 0.5$).

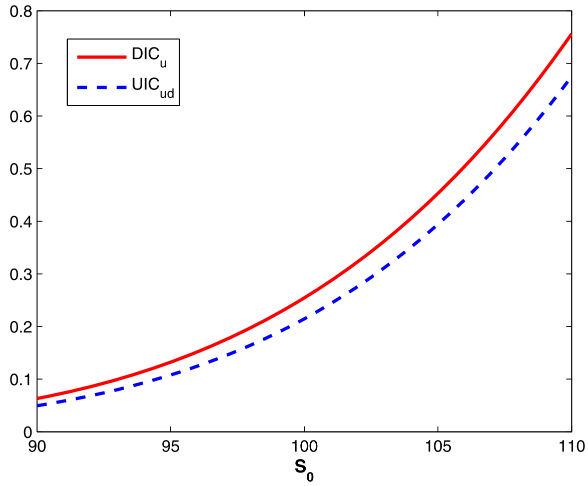

Fig. 5. Comparison between UIC_{ud} and DIC_u, varying S_0 (option parameters: $U = 110$, $D = 90$, $K = 100$, $r = 0.05$, $\sigma = 0.3$, and $T = 0.5$).

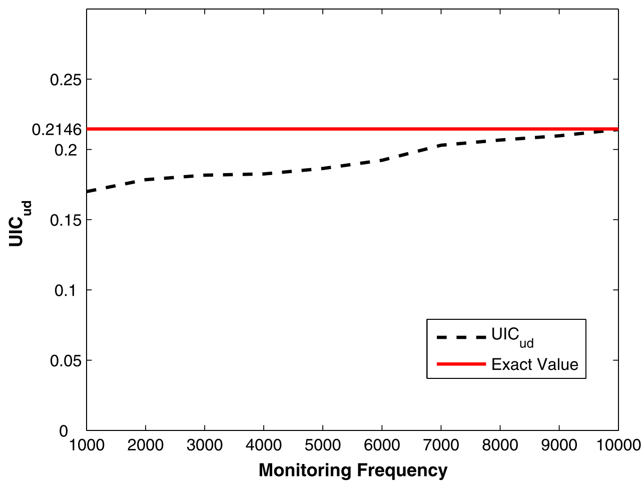

Fig. 6. Monte Carlo simulation results of UIC_{ud} using Antithetic Variates method when monitoring frequency is increased from 1000 to 10,000 ($S_0 = 100$, $K = 100$, $U = 110$, $D = 90$, $r = 0.05$, $\sigma = 0.3$, $T = 0.5$, sample paths 10,000, exact value $= 0.2146$ and the value of UIC_{ud} is 0.2142 when monitoring frequency is 10,000).

is checked at discrete instants through simulations. In fact, the barrier might have been crossed without being detected. To overcome such difficulties, we applied the Antithetic Variates, a Variance Reduction Method of Monte Carlo Method, and simulated the monitoring frequency until 10,000.

5. Conclusion

In this paper, we derived closed-form valuation formulas for barrier options of a new type where the underlying asset price should cross a specified barrier level to activate a regular barrier option. These options are popular in the over-the-counter equity and foreign exchange derivative markets. We further derived explicit valuation formulas for barrier options activated at time the underlying asset price first crosses two barrier levels in a specified order. Due to this contribution, one can price various knock-in and knock-out options of this type. We applied the reflection principle repeatedly on the barrier crossing times in a proper way. A great advantage of our methodology is that it can be easily applied to the case of more complicated structure where further crossing events are chained to activate a barrier option. We also presented the graphs illustrating the properties of the prices, and showed simulation results to confirm the accuracy of our solution.

Acknowledgments

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Appendix A

A.1. Suppose $K < U$. The knock-in call option value at time 0, UIC_d, which is activated at time $\tau = \min\{t\colon\ S_t = D,\ D < S_0\}$ is

$$UIC_d = S_0\left[\left(\frac{D}{S_0}\right)^{\frac{2\tilde{\mu}}{\sigma^2}} N(z_8) + \left(\frac{U}{D}\right)^{\frac{2\tilde{\mu}}{\sigma^2}} N(z_9) - \left(\frac{U}{D}\right)^{\frac{2\tilde{\mu}}{\sigma^2}} N(z_{10})\right]$$
$$- e^{-rT}K\left[\left(\frac{D}{S_0}\right)^{\frac{2\bar{\mu}}{\sigma^2}} N(z_8 - \sigma\sqrt{T}) + \left(\frac{U}{D}\right)^{\frac{2\bar{\mu}}{\sigma^2}} N(z_9 + \sigma\sqrt{T}) - \left(\frac{U}{D}\right)^{\frac{2\bar{\mu}}{\sigma^2}} N(z_{10} + \sigma\sqrt{T})\right]$$

where

$$z_8 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2}{US_0}\right) + \frac{\tilde{\mu}}{\sigma}\sqrt{T}, \qquad z_9 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2}{US_0}\right) - \frac{\tilde{\mu}}{\sigma}\sqrt{T},$$
$$z_{10} = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2K}{U^2S_0}\right) - \frac{\tilde{\mu}}{\sigma}\sqrt{T}.$$

A.2. Suppose $K \geqslant U$. The knock-in call option value at time 0, UIC_d, which is activated at time $\tau = \min\{t\colon\ S_t = D,\ D < S_0\}$ is

$$UIC_d = S_0\left(\frac{D}{S_0}\right)^{\frac{2\tilde{\mu}}{\sigma^2}} N(z_{11}) - e^{-rT}K\left(\frac{D}{S_0}\right)^{\frac{2\bar{\mu}}{\sigma^2}} N(z_{11} - \sigma\sqrt{T})$$

where

$$z_{11} = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^2}{KS_0}\right) + \frac{\tilde{\mu}}{\sigma}\sqrt{T}.$$

A.3. The knock-out call option value at time 0, UOC_d, which is activated at time $\tau = \min\{t\colon\ S_t = D,\ D < S_0\}$ is given by

$$UOC_d = DIC - UIC_d.$$

Appendix B

B.1. Suppose $K > D$. The knock-in call option value at time 0, DIC_{du}, which is activated at time $\tau_2 = \min\{t > \tau_1\colon\ S_t = U,\ \tau_1 = \min\{t > 0\colon\ S_t = D,\ D < S_0\}\}$ is

$$DIC_{du} = S_0\left(\frac{D^2}{US_0}\right)^{\frac{2\tilde{\mu}}{\sigma^2}} N(z_{12}) - e^{-rT}K\left(\frac{D^2}{US_0}\right)^{\frac{2\bar{\mu}}{\sigma^2}} N(z_{12} - \sigma\sqrt{T})$$

where

$$z_{12} = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{D^4}{U^2KS_0}\right) + \frac{\tilde{\mu}}{\sigma}\sqrt{T}.$$

B.2. Suppose $K \leqslant D$. The knock-in call option value at time 0, DIC_{du}, which is activated at time $\tau_2 = \min\{t > \tau_1\colon\ S_t = U,\ \tau_1 = \min\{t > 0\colon\ S_t = D,\ D < S_0\}\}$ is

$$DIC_{du} = S_0\left[\left(\frac{D^2}{US_0}\right)^{\frac{2\widetilde{\mu}}{\sigma^2}} N(z_{13}) + \left(\frac{U}{D}\right)^{\frac{2\widetilde{\mu}}{\sigma^2}} N(z_{14}) - \left(\frac{U}{D}\right)^{\frac{2\widetilde{\mu}}{\sigma^2}} N(z_{10})\right]$$
$$- e^{-rT} K\left[\left(\frac{D^2}{US_0}\right)^{\frac{2\overline{\mu}}{\sigma^2}} N(z_{13} - \sigma\sqrt{T}) + \left(\frac{U}{D}\right)^{\frac{2\overline{\mu}}{\sigma^2}} N(z_{14} + \sigma\sqrt{T}) - \left(\frac{U}{D}\right)^{\frac{2\overline{\mu}}{\sigma^2}} N(z_{10} + \sigma\sqrt{T})\right]$$

where

$$z_{13} = \frac{1}{\sigma\sqrt{T}} \ln\left(\frac{D^3}{U^2 S_0}\right) + \frac{\widetilde{\mu}}{\sigma}\sqrt{T}, \qquad z_{14} = \frac{1}{\sigma\sqrt{T}} \ln\left(\frac{D^3}{U^2 S_0}\right) - \frac{\widetilde{\mu}}{\sigma}\sqrt{T}.$$

B.3. The knock-out call option value at time 0, DOC_{du}, which is activated at time $\tau_2 = \min\{t > \tau_1\colon\ S_t = U,\ \tau_1 = \min\{t > 0\colon\ S_t = D,\ D < S_0\}\}$ is obtained as follows:

$$DOC_{du} = UIC_d - DIC_{du}.$$

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