

Liquidity Risk with Coherent Risk Measures

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ABSTRACT *This paper concerns questions related to the regulation of liquidity risk, and proposes a definition of an acceptable portfolio. Because the concern is with risk management, the paper considers processes under the physical (rather than the martingale) measure. Basically, a portfolio is ‘acceptable’ provided there is a trading strategy (satisfying some limitations on market liquidity) which, at some fixed date in the future, produces a cash-only position, (possibly) having positive future cash flows, which is required to satisfy a ‘convex risk measure constraint’.*

KEY WORDS: Coherent risk measures, liquidity risk, acceptable portfolio

Introduction

We consider questions related to the regulation of liquidity risk. Basically, the firm should be able to unwind its current position without too much loss of its wealth if it were required to do so. Liquidity risk is important in deciding whether a firm’s position is ‘acceptable’ or not. In this paper, we develop a method to incorporate liquidity risk into risk measurement.

Arbitrage pricing theory depends on the existence of an equivalent martingale measure to price securities. Risk management, concerning ‘real’ probabilities of various unfavourable outcomes, is concerned instead with the physical measure.

The notion of acceptability of a random variable was defined in Artzner *et al.* (1999). Carr *et al.* (2001) introduced valuation measures and stress measures instead of ‘generalized scenarios’, and floors associated with probability measures in order to determine whether or not an opportunity is acceptable. Föllmer and Schied (2002a) further developed these ideas. In these papers, a random variable is considered acceptable if its expected value under each scenario measure is greater than equal to a ‘floor’ associated with that measure.

Artzner *et al.* (2002, 2004) discussed multiperiod risks and developed coherent dynamic risk measures on stochastic processes rather than random variables. Larsen

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et al. (2003) and Föllmer and Schied (2002b) considered the set of random variables from which it is possible, by trading, to be acceptable at the terminal date.

Liquidity risk has been described in different ways by several authors. Duffie and Ziegler (2003) modelled the bid–ask spread as a stochastic process correlated with the mid-price process assumed to be a geometric Brownian motion. Çetin *et al.* (2004) modeled a security’s price process as a supply curve (a function of trade size), and then generalized arbitrage pricing theory in their new setting. Baum and Bank (2004) also introduced an illiquid financial market model where a single large trader can move market prices. Longstaff (2001) considered the optimal portfolio choices in an illiquid market where the trading strategies were assumed to be of bounded variation. Some of the microstructure literature provided approaches to issues related to market liquidity (see, for example, O’Hara (2001)).

The paper is organized as follows. In the following section, we give our definition of an acceptable portfolio. We consider a portfolio to be *acceptable* if it can (by trading) be turned into an ‘acceptable’ cash-only position having positive future cash flows at some fixed date. In the third section, we present an example of modelling liquidity. In this example, we require trading strategies to satisfy some limitations on the rates of trading. In the fourth section, we discuss the case in which a portfolio must be liquidated by (finite) random time. We show a method for liquidating a portfolio consisting of a long stock position and a (large, negative) cash position. We obtain the surprising result that we are able to liquidate this position at some stopping time (finite a.s.) for any initial cash position.

Acceptability of Portfolios

In this section, we develop the notion of acceptable portfolios. By a portfolio we mean a collection of securities which provides a stream of future cash flows. We consider a market which consists of a risk-free asset and traded risky assets, S^1 , S^2 , \dots , S^N , which are assumed to be adapted stochastic processes on a given probability space (Ω, \mathcal{F}, P) with a filtration.

A trading strategy $\pi_t = (\pi_t^0, \pi_t^1, \dots, \pi_t^N)$ is any $(N+1)$ -dimensional $\{\mathcal{F}_t\}$ -adapted process: π_t^0 is the number of units of the risk-free asset (the amount of cash holding) and π_t^n is the number of shares of asset S^n ($n=1, \dots, N$) held at each time t .

In this section, we won’t specify the restrictions on trading or the cost of trading. We will instead work with the more general framework; trading occurs as usual, but may be limited by market liquidity, and may be subject to further liquidity-based restrictions (for example, traders can’t sell ‘too fast’.)

We consider the ‘mark-to-market’ value for a portfolio which is evaluated at ‘quoted’ prices as in the classical pricing theory. Although a firm cannot obtain this amount of cash by immediate liquidation, we shall call this value simply the *wealth*. We assume that for ‘admissible’ trading strategies (to be defined in order to fit market restrictions) no additional cashflows besides trading (such as dividends) is generated, and the wealth (‘mark-to-market’ value) is bounded below.

We begin with a convex measure of risk (as in Föllmer and Schied (2002a)) defined by a set of scenario measures $\{P^i, i \in I\}$ and associated floors $f^i \in \mathbb{R}$. Recall that a random variable is considered acceptable if its expected value under each scenario measure is greater than equal to the floor associated with that measure.

Definition 1. A random variable V is *acceptable* if

$$E_{p_i}[V] \geq f^i \quad \text{for all } i \in I \tag{1}$$

We first define a positive portfolio as follows:

Definition 2. A portfolio is said to be *positive* if it entails only non-negative cashflows in the future.

Now we define a portfolio to be acceptable provided it can be liquidated by some date T into a cash-only position whose (discounted) value is acceptable in the sense of Definition 1, and (possibly) having additional positive cashflows in the future.

More precisely:

Definition 3. A portfolio X is *acceptable* if there exist an ‘admissible’ trading strategy π_t and a date T such that X can be decomposed (by trading) into a cash-only position C and a positive portfolio by date T . That is,

- (i) $\pi_T^{n,C} = 0$ for all $1 \leq n \leq N$ where $\pi_T^{n,C}$ denotes the number of shares (corresponding to cash-only part C) of asset S^n held at date T , and
- (ii) the random variable $e^{-rT} \pi_T^{0,C}$ satisfies (1), which means that the discounted value of cash-only part C is acceptable.

We denote by \mathcal{A} the acceptance set, i.e. the set of all acceptable portfolios. The acceptance set \mathcal{A} has the following property:

Proposition 1. The acceptance set \mathcal{A} is convex; i.e. if X is acceptable and Y is acceptable, then so is $\lambda X + (1 - \lambda)Y$ for $0 \leq \lambda \leq 1$.

Proof. Since X is acceptable, there exists an admissible trading strategy ϕ_t for X which satisfies $\phi_{T_1}^{n,C} = 0$ for every $1 \leq n \leq N$ and $e^{-rT_1} \phi_{T_1}^{0,C}$ is acceptable for some T_1 . Since Y is also acceptable, there exists an admissible trading strategy ψ_t for Y such that $\psi_{T_2}^{n,C} = 0$ for all $1 \leq n \leq N$ and $e^{-rT_2} \psi_{T_2}^{0,C}$ is acceptable for some T_2 .

Set $\bar{T} = \max\{T_1, T_2\}$. Without loss of generality, letting $T = T_1$, take the trading strategy $\pi_t = \lambda \phi_t + (1 - \lambda)\psi_{t \wedge T_2}$ where $\psi_{t \wedge T_2}$ is defined in an obvious way. Then, for this strategy π_t the portfolio $\lambda X + (1 - \lambda)Y$ is decomposed into a cash-only position and a positive portfolio by date T , and the discounted value of cash-only part is

$$\lambda e^{-rT_1} \phi_{T_1}^{0,C} + (1 - \lambda) e^{-rT_1} e^{r(T_1 - T_2)} \psi_{T_2}^{0,C}$$

which is an acceptable random variable.

Remark 1. We observe that the acceptance set \mathcal{A} is monotone in the sense that if X is acceptable and $X \leq Y$ ($Y - X$ produces non-negative cash flows), then Y is acceptable. Since X is acceptable, there exist an admissible trading strategy ϕ_t and a date T for X . Then for the trading strategy ϕ_t , Y is liquidated at date T with additional positive portfolio $e^{rT}(Y - X)$.

Remark 2. The acceptance set \mathcal{A} is *not necessarily* positively homogeneous. Because for an acceptable portfolio X , there is no guarantee that there will be an admissible trading strategy to liquidate the doubled portfolio $2X$ (because of the restrictions on trading). This can be interpreted as the increase of liquidity risk for large positions.

The Model

This section introduces an example of modelling liquidity risk. We consider a market which consists of a risk-free asset and one risky asset S on a given probability space (Ω, \mathcal{F}, P) with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$.

The price of the risky asset follows an $\{\mathcal{F}_t\}$ -adapted geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

and the price of the risk-free asset (the amount of cash in a bank account) grows at the interest rate r .

In a market with friction (for example bid–ask spreads), the market provides different prices for buying and selling stock. For simplicity, the actual price traded in the market can be modelled by the following:

$$P_t^\pm = S_t \pm \frac{\lambda}{2} S_t = \left(1 \pm \frac{\lambda}{2}\right) S_t$$

Where P^+ , P^- represent the prices for buyers and for sellers respectively, and λ indicates the bid–ask spread. We may think that a trader pays proportional transaction costs of $\frac{\lambda}{2}$ in trading.

We define the set of trading strategies to be the set of all $\{\mathcal{F}_t\}$ -adapted processes with left continuous paths that have right limits. We identify an element π_t of the set with a vector stochastic process (π_t^0, π_t^1) where π_t^0 denotes the amount held in cash and π_t^1 the number of shares of asset S held at time t .

We restrict the set of trading strategies available to a firm by the condition that the firm cannot ‘liquidate’ too fast: We shall assume that a trading strategy is allowed if the changes in the number of shares of asset S held over any time interval never exceed ε -multiple of the length of the time interval. We note that ε might be determined by market conditions such as the daily trading volume of the asset. It might also be given in advance through the negotiations between the firm and the supervisor.

We define the set of *admissible trading strategies* as follows:

Definition 4. A trading strategy π_t is *admissible* if it satisfies

$$|\pi_{t_1}^1 - \pi_{t_2}^1| \leq \varepsilon |t_1 - t_2| \quad \text{for all } t_1, t_2 \geq 0$$

and keeps the wealth (‘mark-to-market’ value) bounded below.

This ensures that a firm cannot take advantage of certain pathological varieties of arbitrage, such as doubling strategies. For an admissible trading strategy π_t , since π_t^1 is Lipschitz continuous in t a.s., π_t^1 is a process of boundedvariation. Thus there is a minimal representation for π_t^1 with some pair of $\{\mathcal{F}_t\}$ -adapted, increasing processes

(Π_t^+, Π_t^-) such that

$$\pi_t^1 = \Pi_t^+ - \Pi_t^-$$

In fact, Π_t^+ is interpreted as the cumulative number of shares of asset S bought and Π_t^- as the cumulative number sold until time t for the strategy π_t . Then π_t^0 , the value of cash holding at time t , can be described as

$$\begin{aligned} d\pi_t^0 &= r\pi_t^0 dt - d(\Pi_t^+)P_t^+ + d(\Pi_t^-)P_t^- \\ &= r\pi_t^0 dt - (d\pi_t^1)S_t - \frac{\lambda}{2}(d(\Pi_t^+) + d(\Pi_t^-))S_t \end{aligned} \quad (2)$$

The first term on the right-hand side is due to interest, and the terms containing S_t are caused by trading. We assume that admissible trading strategies are self-financing, that is, besides trading no additional cashflows (such as dividends) will be generated.

Definition 5. A trading strategy $\pi_t = (\pi_t^0, \pi_t^1)$ is said to be *self-financing* if it satisfies the equation

$$d(\pi_t^0 + \pi_t^1 S_t) = r\pi_t^0 dt + \pi_t^1 dS_t - \frac{\lambda}{2}(d(\Pi_t^+) + d(\Pi_t^-))S_t$$

We consider a set of test measures $\{P^i, i \in I\}$ and associated floors $f^i \in \mathbb{R}$, where each test measure is absolutely continuous with respect to P and $\tilde{f} = \max\{f^i, i \in I\}$ is bounded. We denote by \mathcal{Q} the set of probability measures absolutely continuous with respect to P , under which the (discounted) asset price process is a local martingale. We shall assume that the set of test measures has a nonempty intersection with \mathcal{Q} as in Artzner et al. (1999, Condition 4.3), i.e. $P^i = Q$ for some i and $Q \in \mathcal{Q}$.

Lemma 1. The discounted wealth process is a supermartingale under any $Q \in \mathcal{Q}$.

Proof. The wealth ('mark-to-market' value) process W is written as

$$\begin{aligned} dW(t) &= r\pi_t^0 dt + \pi_t^1 dS_t - \frac{\lambda}{2}(d(\Pi_t^+) + d(\Pi_t^-))S_t \\ &= r\{W(t) - \pi_t^1 S_t\} dt + \pi_t^1 dS_t - \frac{\lambda}{2}(d(\Pi_t^+) + d(\Pi_t^-))S_t \end{aligned}$$

Then

$$d(e^{-rt} W(t)) = e^{-rt} \left\{ -r\pi_t^1 S_t dt + \pi_t^1 dS_t - \frac{\lambda}{2}(d(\Pi_t^+) + d(\Pi_t^-))S_t \right\}$$

thus

$$e^{-rt}W(t) = W(0) + \int_0^t \pi_u^1 d(e^{-ru}S_u) - \frac{\lambda}{2} \int_0^t e^{-ru}S_u (d(\Pi_t^+) + d(\Pi_t^-))$$

The stochastic integral (with respect to $e^{-n}S_t$), the second term on the right-hand side, is a local martingale (Protter, 2004, p.128). We note that a continuous local martingale which is bounded below is a supermartingale (Karatzas and Shreve, 1988). Since the last term (without the minus sign) is non-negative and non-decreasing, we conclude that the discounted wealth process $e^{-rt}W(t)$ is a supermartingale for every $Q \in \mathcal{Q}$.

Proposition 2. For any fixed date T , there is a constant K such that if the initial ‘mark-to-market’ value of a portfolio is less than K , then the portfolio cannot be decomposed into an acceptable cash-only position and a positive portfolio by time T , thus the portfolio is not acceptable.

Proof. Suppose a portfolio X is acceptable, that is, X is decomposed into a cash-only position whose discounted value is an acceptable random variable and a positive portfolio by some date T . Then, there must exist an admissible trading strategy π_t for which $\pi_T^{1,C} = 0$ and $e^{-rT}\pi_T^{0,C}$ is acceptable, i.e.

$$E_{P^i} \left[e^{-rT} \pi_T^{0,C} \right] \geq f^i$$

for every test measure P^i .

On the other hand, if test measure P^i belongs to \mathcal{Q} , the discounted wealth process is a supermartingale under P^i by Lemma 1. Then

$$\begin{aligned} E_{P^i} \left[e^{-rT} \pi_T^{0,C} \right] &\leq E_{P^i} \left[e^{-rT} W(T) \right] \\ &\leq W(0) \end{aligned} \tag{3}$$

Let $K = \max \{f^i: P^i \in \mathcal{Q}\}$, the maximum value of floors associated with $P^i \in \mathcal{Q}$. If the initial ‘mark-to-market’ value $W(0)$ is less than K , then there is no admissible trading strategy for which the portfolio is liquidated into an acceptable cash-only position by date T . In other words, there is no acceptable way to liquidate the position. The portfolio cannot be acceptable.

Remark 3. Proposition 2 remains true when a fixed date T is replaced by a bounded stopping time. In other words, for any bounded stopping time τ , there is a constant such that if the initial ‘mark-to-market’ value of a portfolio is less than that constant, then the portfolio cannot be liquidated into an acceptable cash-only position (having a positive future cash flows) by random time τ , because, the last inequality in (3) is preserved by the optional sampling theorem (Ikeda and Watanabe, 1989, p. 26).

Remark 4. We can obtain the result of Proposition 2 when the condition that the set of test measures has a non-empty intersection with \mathcal{Q} is relaxed to that the convex hull of $\{P^i, i \in I\}$ has a non-empty intersection with \mathcal{Q} , i.e. there is a convex combination of test measures $\sum_i \alpha_i P^i$ ($\sum_i \alpha_i = 1$ and $\alpha_i \geq 0$ for all i) which belongs to \mathcal{Q} .

Liquidation for Long Position in (Almost Surely) Finite Time

In this section, we consider a possible relaxation of definition of an acceptable portfolio to permit trading up to a random (rather than constant) time to obtain a cash-only position and a positive portfolio. To discuss this question, we present a liquidation method for a long position when the risky asset grows fast enough (suppose $\mu > r + \frac{\sigma^2}{2}$.) We obtain the interesting result that we can liquidate any long stock position into an acceptable cash-only position, having positive future cash flows, in (almost surely) finite time for every (even large negative) initial cash holdings. We note that under the requirement of finite fixed time for liquidation as in Definition 3, this cannot happen by Proposition 2.

The stock price process follows

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

Without loss of generality (using a change of numeraire), we may assume the interest rate $r=0$. We assume that the initial stock price $S_0=1$ and initial shares of stock held $\pi_0^1=1$. We use the model as in the previous section, i.e. we incorporate liquidity constraints by assuming that a trader can't sell stock at a rate faster than rate ε .

Construct a trading strategy as follows:

Step 1. Hold the stock until the stock holdings are worth $L (\gg 1)$.

Here we show that the time (denoted by σ_1 in the following) at which the value of stock holdings reaches L is finite a.s., and has a finite expected value. Since $\pi_t^1 = \pi_0^1 = 1$ for time $t < \sigma_1$, the value of stock holding is S_t . Set

$$\begin{aligned} \sigma_1 &= \inf \{ t : S_t = L \} \\ &= \inf \left\{ t : \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma B_t = \ln L \right\} \end{aligned}$$

From the fact that $\mu - \frac{\sigma^2}{2} > 0$ and the Brownian hitting time

$$\inf \{ t : \sigma B_t = \ln L \}$$

is finite a.s., we have $\sigma_1 < \infty$ a.s..

Let $Z_t = \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma B_t$. Since Z_t is a Brownian motion with a positive drift under P , following Karlin and Taylor (1975, p. 362),

$$E_P [e^{-\theta \sigma_1}] = \exp \left\{ - \frac{\ln L}{\sigma^2} \left(\sqrt{\left(\mu - \frac{\sigma^2}{2} \right)^2 + 2\sigma^2 \theta} - \left(\mu - \frac{\sigma^2}{2} \right) \right) \right\}$$

for any fixed $\theta > 0$. Differentiating the formula with respect to θ and letting $\theta \rightarrow 0$, we obtain

$$E_P[\sigma_1] = \frac{1}{\mu - \frac{\sigma^2}{2}} \ln L.$$

Step 2. Sell the stock at rate ε and stop when the stock holdings are worth 1. We will first show that the time τ_1 at which the value of stock holding decreases (by trading) to 1 is finite a.s. Set

$$\begin{aligned} \tau_1 &= \inf\{t \geq \sigma_1 : \pi_t^1 S_t = 1\} \\ &= \inf\{t \geq \sigma_1 : (1 - \varepsilon(t - \sigma_1)) S_t = 1\} \end{aligned}$$

Note that $\tau_1 < \sigma_1 + \frac{1}{\varepsilon}$, and at time τ_1 , the firm owns $\pi_{\tau_1}^1 = 1 - \varepsilon(\tau_1 - \sigma_1) < 1$ shares of stock.

Then the change in the value of cash holding is, following the equation (2),

$$\begin{aligned} d\pi_t^0 &= -d(\Pi_t^+) P_t^+ + d(\Pi_t^-) P_t^- \\ &= -(d\pi_t^1) S_t - \frac{\lambda}{2} (d(\Pi_t^+) + d(\Pi_t^-)) S_t \\ &= \varepsilon S_t - \frac{\lambda}{2} \varepsilon S_t \\ &= \varepsilon \left(1 - \frac{\lambda}{2}\right) S_t \end{aligned}$$

for $\sigma_1 < t < \tau_1$. Thus the total amount of changes in the value of cash holding (by selling the stock) for the period between σ_1 and τ_1 is

$$Y_1 = \int_{\sigma_1}^{\tau_1} d\pi_t^0 = \varepsilon \left(1 - \frac{\lambda}{2}\right) \int_{\sigma_1}^{\tau_1} S_t dt$$

Step 3. Wait until the stock holdings are worth L .

First we consider the distribution of process $\frac{S_t}{S_{\tau_1}} (t \geq \tau_1)$ for time $t \geq \tau_1$. For $t \geq \tau_1$, the stock price process follows

$$S_t = S_{\tau_1} \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)(t - \tau_1) + \sigma(B_t - B_{\tau_1})\right\}$$

By the strong Markov property of Brownian motion, $B_t - B_{\tau_1}$ is a Brownian motion, which is independent of \mathcal{F}_{τ_1} . Then $\frac{S_t}{S_{\tau_1}} (t \geq \tau_1)$ is independent of \mathcal{F}_{τ_1} and the distribution of $\frac{S_t}{S_{\tau_1}}$ conditioned on \mathcal{F}_{τ_1} is the same as the distribution of S_u , where $u = t - \tau_1$ denotes the amount of time since τ_1 .

Now let

$$\begin{aligned}\sigma_2 &= \inf \left\{ t \geq \tau_1 : \pi_{\tau_1}^1 S_t = L \right\} \\ &= \inf \left\{ t \geq \tau_1 : \frac{S_t}{S_{\tau_1}} = L \right\}\end{aligned}$$

since $\pi_{\tau_1}^1 S_{\tau_1} = 1$ at time τ_1 . Therefore, the time σ_2 at which the value of stock holding increases up to L is finite a.s. by the same reasoning as in Step 1.

Step 4. Sell the stock at rate $\frac{\varepsilon}{S_{\tau_1}}$ and stop when the stock holdings are worth 1. We note that $\frac{\varepsilon}{S_{\tau_1}}$ satisfies the restriction on the rate of trading. Because $\frac{\varepsilon}{S_{\tau_1}} = \pi_{\tau_1}^1 \varepsilon < \varepsilon$. Set

$$\begin{aligned}\tau_2 &= \inf \left\{ t \geq \sigma_2 : \pi_t^1 S_t = 1 \right\} \\ &= \inf \left\{ t \geq \sigma_2 : \left(\pi_{\tau_1}^1 - \frac{\varepsilon}{S_{\tau_1}}(t - \sigma_2) \right) S_t = 1 \right\} \\ &= \inf \left\{ t \geq \sigma_2 : (1 - \varepsilon(t - \sigma_2)) \frac{S_t}{S_{\tau_1}} = 1 \right\}\end{aligned}$$

Note that $\tau_2 < \sigma_2 + \frac{1}{\varepsilon}$, and at time τ_2 , the firm owns $\pi_{\tau_2}^1 = \frac{1}{S_{\tau_1}}(1 - \varepsilon(\tau_2 - \sigma_2)) < \frac{1}{S_{\tau_1}} = \pi_{\tau_1}^1$ shares of stock.

Then, for $\sigma_2 < t < \tau_2$

$$\begin{aligned}d\pi_t^0 &= -d(\Pi_t^+)P_t^+ + d(\Pi_t^-)P_t^- \\ &= \frac{\varepsilon}{S_{\tau_1}} \left(1 - \frac{\lambda}{2}\right) S_t\end{aligned}$$

Thus, the amount transferred into cash holdings (by selling the stock) for the period between σ_2 and τ_2 is

$$Y_2 = \int_{\sigma_2}^{\tau_2} d\pi_t^0 = \varepsilon \left(1 - \frac{\lambda}{2}\right) \int_{\sigma_2}^{\tau_2} \frac{S_t}{S_{\tau_1}} dt$$

Step 5. Repeat the process to produce Y_3, Y_4, \dots

Now we will show that there is some time point at which all the risk constraints are satisfied. We consider a set of test measures $\{P^i, i \in I\}$ and associated floors $f^i \in \mathbb{R}$ as before. In an infinite time horizon, each test measure is assumed to be ‘locally’ absolutely continuous with respect to P , in the sense that $P^i|_{\mathcal{F}_t}$ (the restriction of P^i to (Ω, \mathcal{F}_t)) is absolutely continuous with respect to $P|_{\mathcal{F}_t}$ (the restriction of P to (Ω, \mathcal{F}_t)) for every $0 < t < \infty$.

Proposition 3. Assume that $\mu > \frac{\sigma^2}{2}$ ($r=0$). Let X be a portfolio whose initial shares of stock held is 1. Then there exist an admissible trading strategy and a (finite) stopping time τ^* such that X is decomposed into a cash-only position C and a

positive portfolio by date τ^* , and (discounted) value of cash-only part C is an acceptable random variable, i.e.

$$E_{P^i} \left[\pi_{\tau^*}^{0,C} \right] \geq f^i \quad \text{for all } i \in I$$

Therefore for every (even large negative) initial cash holdings, a long stock position can be liquidated into an acceptable cash position in almost surely finite time.

Proof. Use the trading strategy described above. Consider

$$Y_1 = \varepsilon \left(1 - \frac{\lambda}{2} \right) \int_{\sigma_1}^{\tau_1} S_t dt \quad Y_2 = \varepsilon \left(1 - \frac{\lambda}{2} \right) \int_{\sigma_2}^{\tau_2} \frac{S_t}{S_{\tau_1}} dt$$

By the strong Markov property, the process $\frac{S_{\tau_1+u}}{S_{\tau_1}} (u \geq 0)$ is a geometric Brownian motion starting at 1, independent of \mathcal{F}_{τ_1} , and having the same distribution as $S_t (t \geq 0)$. Considering the definitions of $\sigma_1, \tau_1, \sigma_2,$ and τ_2 , we observe that the random variables Y_1 and Y_2 are independent and identically distributed.

In the same way, we have a sequence of random variables Y_1, Y_2, \dots , which are independent, identically distributed, and have a positive mean. By the law of large numbers,

$$Y_1 + Y_2 + \dots \rightarrow \infty \text{ a.s.}$$

under P . Consider the process Z_t

$$Z_t = \pi_0^0 + \sum_{\tau_m \leq t} Y_m$$

where π_0^0 is the amount of initial cash holdings. We then have $Z_t \leq \pi_0^0$ and $Z_t \rightarrow \infty$ a.s. under P .

Let $\tau^* = \inf \{ t : Z_t \geq 2\bar{f} \}$ where $\bar{f} = \max \{ f^i, i \in I \}$. We note that $\tau^* < \infty$ a.s. and

$$A = \{ \omega : Z_{\tau^*} < \bar{f} \} \in \mathcal{F}_{\tau^*}$$

is a P -null set.

Since each P^i is locally absolutely continuous with respect to P , there exists an increasing sequence $\{T_n\}$ of stopping times such that $P^i \{ \lim T_n = \infty \} = 1$ and $P^i|_{\mathcal{F}_{T_n}}$ is absolutely continuous with respect to $P|_{\mathcal{F}_{T_n}}$ for all n (Jacod and Shiryaev, 1987, p. 153). For the localizing sequence $\{T_n\}$ (depending on P^i),

$$P^i \{ Z_{\tau^*} < \bar{f} \} = \lim_{n \rightarrow \infty} P^i \{ \{ Z_{\tau^*} < \bar{f} \} \cap \{ T_n \geq \tau^* \} \} = 0$$

because $\{ Z_{\tau^*} < \bar{f} \} \cap \{ T_n \geq \tau^* \}$ is \mathcal{F}_{T_n} -measurable (Jacod and Shiryaev, 1987, p. 4) and a P -null set. Then

$$E_{P^i} [\pi_{\tau^*}^0] \geq E_{P^i} [Z_{\tau^*}] \geq \bar{f} \geq f^i$$

for every $i \in I$. Therefore, at random time τ^* (finite a.s.), the value of cash-only part

$\pi_{\tau^*}^{0,C}$ is an acceptable random variable, and $\pi_{\tau^*}^1 (>0)$ shares of stock (positive portfolio) remains.

From the observation in this section, we conclude that the requirement of finite fixed time (or, possibly bounded random time) for liquidation is necessary in the regulation of liquidity risk.

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