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Valuation of European options in the market with daily price limit

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A valuation problem of the European style contingent claim in the market with daily price movement limit is studied. Unlike the one leading to the well known Black–Scholes formula, this problem depicts considerable conceptual difficulty and anomaly created by the presence of various arbitrage opportunities inherently built in the model due to the daily price movement limit. The presence of arbitrage makes it go against the grain of the well established arbitrage pricing theory. In this paper, how these complications arise are discussed and then a valuation approach devised, which is called the 'vanishing transaction cost technique,' of getting around the difficulty.

Keywords: geometric Brownian motion with boundary, slowly reflecting boundary, arbitrage, Black–Scholes formula, vanishing transaction cost technique

1. Introduction

The basic premise of Black and Scholes (1973) option pricing is that the process S_t representing the stock price obeys the simple diffusion process defined by the following stochastic differential equation

$$dS_t/S_t = \mu dt + \sigma dB_t,$$

where B_t is the standard Brownian motion, and μ and σ are constants.

Several models to relax the above assumptions are proposed and studied by many authors. Some authors consider, for example, the processes with stochastic volatility or jump, and some study the effects of the transaction costs, or the incompleteness of the market, or the trading restriction. However, all of them subscribe to the assumptions that the process is Markov and the arbitrage cannot occur. They seem like very natural assumptions in view of the economic principle and the Efficient Market Hypothesis.

However, in some markets, such assumptions cannot be made without examining carefully the restrictions governing the behaviour of the market. One such example is the market with the daily

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price movement limit, which means that the price of an individual stock in each trading day cannot move above or below a certain fixed percentage point of the closing price of the previous trading day. That means that the stock price process in each trading day has the movement range, and the movement range of the next day is determined as a fixed percentage point above or below the closing price of the present trading day. Since the closing price is stochastic, so is the trading range of the next day. There are many markets which adopt the daily price movement limit rule: The Korean stock market has the daily price movement limit of 8%, and the Japanese stock market has similar rule, but with wider range, and so does the commodity futures market.

In Section 2, we propose a stochastic process model which is a direct analogue in this setting of the well known log-normal diffusion process. It is done in two steps. First, the price process of a given day can be modelled using the geometric Brownian motion with boundary. Once the price hits the upper or lower limit, it may stay at that level all day, or, as the market condition dictates, it may move inside the trading range later in the same trading day. So its boundary behaviour resembles the geometric Brownian motion whose boundary is slowly reflecting. For lack of terminology, we call such process a geometric Brownian motion with boundary. It satisfies a stochastic differential equation with boundary condition, and is studied in Section 2.1. The second step is to extend the process to the multiday period. The simplest and the most natural assumption to make is that the next day is just like any other day except that it has a different trading range. It is studied in Section 2.2, and it is proved that this kind of periodicity condition essentially implies that such process cannot be Markov.

Next, we study the valuation problem of the European style contingent claim in this kind of market. Before we go on, let us make one more assumption: namely, the trader can trade at the prevailing price at any time of the trading day.

Examine now carefully various mathematical possibilities of arbitrage opportunities associated with this kind of process assuming there are no transaction costs. Suppose the price is at the upper limit in the middle of trading day. Then the trader can employ the trading strategy of selling short the stock at that price and covering back at the end of the trading day. As the price is guaranteed not to go over the upper limit, the trader will not lose as long as he or she covers it back before the end of the day, and the trader gains in two ways: first, if he or she may be able to cover it back at a lower price, the difference is his or her profit; and as the interest is compounded continuously, the trader will get a sure profit, albeit minuscule, due to the interest, even if he or she has to cover it back at the same price. If the price is at the lower limit in the middle of the trading day, the same argument applies. This anomaly can be interpreted as a weak form of arbitrage because the trader will not lose under any circumstances and the expected profit is positive. Of course, one may argue that it is not significant enough to be taken into consideration in the actual practice. However, this mathematical possibility of arbitrage opportunities will lead into inconsistency of the model, which make it impossible to apply the arbitrage pricing theory that is firmly established in the theory of finance. We also assume that the interest is paid at the beginning of each trading day. But, we use continously compounded interest rate as a proxy in the calculation.

In Section 3, we study a simplistic model with no transaction costs in which no arbitrage is possible. In order to avoid the arbitrage, we assume that the boundary of the geometric Brownian motion with boundary is absorbing and the interest rate is zero. Even in this simplistic case, our result is interesting, since our process is still non-Markov. It also serves as an illustration of our philosophy for the remainder of this paper.

If one wants to consider the case which is more general than that in Section 3, one immediately faces lots of difficulties. Above of all, the aforementioned arbitrages make it very difficult to apply any established arbitrage pricing theory framework. To get around this difficulty, we present a new conceptual devise. It is based on the following observation: The aforementioned theoretical arbitrage cannot be profitably employed in practice largely because of the transaction costs, so the problem may be gotten around by incorporating the 'residual effect' of the transaction costs in the model, even if no actual transaction costs are incurred. One way of incorporating the 'residual effect' of the transaction costs is to use a certain discretization scheme à la Leland (1985), and let the transaction costs vanish sufficiently fast as the size of the discretization interval gets to zero. Thus in the end the transaction costs disappear, and we obtain a certain partial differential equation satisfying certain initial boundary value conditions, which can be used to value the option in this context. This method is reminiscent of the vanishing viscosity technique of solving the Euler equation in fluid mechanics, and in this sense, we call our method a 'vanishing transaction cost technique.'

We would like to thank the referee for suggestions in several places for certain conceptual clarification.

2. The price process

In this section, we describe a new kind of stochastic process which models the price process having the daily price movement limit. When the range of movement is small, it behaves significantly differently from the geometric Brownian motion. We first introduce an intra-day model, which we call the geometric Brownian motion with boundary. We then extend this to a multiday model, which we call the geometric Brownian motion with stochastically moving boundary.

2.1. *Geometric Brownian motion with boundary (intraday model)*

The stochastic process that models the price process in a single day must meet several characteristic requirements. First, it must behave like the geometric Brownian motion with drift when the process is inside the movement range, but its boundary behaviour should be general enough to encompass the slowly reflecting case as well absorbing or instantaneously reflecting one. The simplest and the most natural one is given by the following stochastic differential equation:

$$\begin{cases} dX(t) = \sigma(X(t))I_{(a,b)}(X(t))dB_t + \mu(X(t))I_{(a,b)}(X(t))dt + \delta_1 d\phi_t - \delta_2 d\psi_t \\ I_{\{a\}}dt = \rho_1 d\phi_t \\ I_{\{b\}}dt = \rho_2 d\psi_t \end{cases} \tag{1}$$

where ϕ and ψ are so-called local times at a and b, respectively. The probability space is taken to be $(W, \mathscr{B}(W))$, where W is the space consisting of all continuous functions defined on $[0, \infty]$ and $\mathscr{B}(W)$ is the usual Borel σ-field of W. For the intuitive understanding, one may envision that ρ represents how sticky the boundary is. In fact, $\rho > 0$ indicates that the boundary is not instantaneously reflecting and $\delta > 0$ indicates that the boundary is not absorbing. This stochastic differential equation is well known. For example, one may find the existence and uniqueness proof in the multidimensional case in Ikeda and Watanabe (1989).

Theorem 1

Assume that σ and μ in Equation 1 are continuous on $[a, b]$ and also assume that

$$\sigma(x) \geqslant c$$

$$\delta_1, \delta_2 \geqslant c$$

for some positive constant c. Then for any probability π on $([a, b], \mathscr{B}([a, b]))$, there exists a solution $X(t)$ such that the probability law of $X(0)$ coincides with π. Furthermore, the uniqueness of solutions holds.

Remark

The assumption $\delta_1, \delta_2 \geqslant c$ can be weakened to the condition that $\delta_1 + \rho_1 > 0$ and $\delta_2 + \rho_2 > 0$.

2.2. *Geometric Brownian motion with stochastically moving boundary (multiday model)*

In this subsection we extend the above intraday process to a multiday one. The simplest and the most reasonable assumption to make is that any given day is like any other day except that the range of movement of each day is determined by the closing value of the previous day. This kind of periodicity and path dependency inevitably implies that this new process cannot be Markov, and this causes some complication in the economic behaviour of the model. Let $X(t)$ be the process starting at $X(0) = x_0$ and $[(n-1)T, nT]$ be the time interval for the nth day.

To be clear, we itemize our assumptions as follows.

- For the nth day, the price process is assumed to be the geometric Brownian motion with boundary as described in Section 2.1 with the movement range $[a, b] = [(1-\alpha)X_{(n-1)T}, (1+\beta)X_{(n-1)T}]$, where α and β are fixed positive constants.
- The closing value of the nth trading day is the same as the starting value of the $n+1$st trading day.
- The process repeats itself in the next day, in other words, the process for any trading day is the same as the process for any other trading day except that the movement range is changed.

Theorem 2

Any process defined to satisfy the above requirements cannot be a continuous time Markov process with the transition probabilities satisfying

(1) $p_{t,\eta}(B|x) = \int p_{t,s}(B|y)p_{s,\eta}(\mathrm{d}y|x)$,

$$\text{for any} \quad x \in [a, b] \text{ and } T \geqslant t > s > \eta \geqslant 0$$

(2) $p_{t,\eta}(\cdot|x) \xrightarrow{\mathscr{D}} \delta_{\{x\}}(\cdot)$

as $t \longrightarrow \eta$ *for any* $x \in [a, b]$

(3) $p_{nT+t,nT}(B + x|x) = p_{t,0}(B + x_0|x_0)$

for $n \in \mathbb{N}$ *and fixed* T

where B *is any Borel set of* $\mathbb{R}$.

Proof

We may assume for simplicity that $\alpha = \beta = r$, $x_0 = 1$. If $X(t)$ is such a Markov process, then its transition probabilities satisfy the following special case of Chapman–Kolmogorov equations

$$p_{T+\eta,T}(B|x) = \int p_{T+\eta,T+s}(B|z)p_{T+s,T}(\mathrm{d}z|x)$$

for every $B \in \mathscr{B}(\mathbb{R})$, $x \in [1 - r, 1 + r]$, $T \geqslant \eta > s \geqslant 0$. Let $B = [(1 + r)^2 - \varepsilon, (1 + r)^2]$ and $x = 1 + r - \varepsilon/(1 + r)$ for any sufficiently small ε. Then $p_{T+\eta,T}(B|x) = 0$ implies $p_{T+\eta,T+s}(B|z) = 0$ for almost every $(1 - r)x < z < (1 + r)x$. Combining Equations 1 and 2, we have $p_{T+\eta,T}(B|z) = 0$. But, by Equation 3, this is a contradiction.

Although the process is non-Markov as a continuous time process, the discrete time process constructed by restricting the continuous time one at the end of each day becomes a Markov chain.

Theorem 3

$Z_n = X_{nT}$, $n = 0, 1, 2 \ldots$, *is a Markov chain, i.e., a discrete time Markov process.*

Proof

Let the one-step transition probabilities in the intraday model be $p(B|x) = P(X(T) \in B|X(0) = x)$ for all $B \in \mathscr{B}(\mathbb{R})$ and initial distribution π. Since our process X has the periodicity property,

$$P(Z_n \in B_n, Z_{n-1} \in B_{n-1}, \ldots, Z_0 \in B_0)$$
$$= \int_{B_{n-1}} \int_{B_{n-2}} \cdots \int_{B_0} p(B_n|x_{n-1})p(\mathrm{d}x_{n-1}|x_{n-2}) \cdots p(\mathrm{d}x_1|x_0)\pi(\mathrm{d}x_0)$$

for all $B_n, \ldots, B_0 \in \mathscr{B}(\mathbb{R})$. It can be shown that a process satisfying this property is a Markov chain with $p(B|x)$ as its transition probabilities.

3. A simple model

As discussed in the Introduction, the market with the daily price limit has intrinsic arbitrage opportunities, which cause logical problems in applying the well-established arbitrage pricing theory that must require no arbitrage assumption. However, one can certainly eliminate the arbitrage opportunities if appropriate restrictions on the model are made. One simple case is to assume that the stochastic process is a geometric Brownian motion with stochastically moving boundary, where the boundary is absorbing, the interest rate is zero and there are no transaction costs. In this section, we present a valuation method under these assumptions. Although this case may be too simplistic, the result in this section nonetheless serves as a guide to our philosophy. Moreover, it is still an interesting result in its own right because the price process for the multiday periods is non-Markovian. It is to be noted that the pair of the daily price limit and the price itself constitutes a two-dimensional Markovian process. However, since the daily price limit is not a tradable quantity, using this two-dimensional Markov process does not seem to make the problem any easier.

3.1. *Intra-day valuation*

Assume that the stock price S_t is a solution of Equation 1 with $\sigma(S_t) = \sigma S_t$ and $\mu(S_t) = \mu S_t$ for some constants σ and μ. Since the boundary is assumed to be absorbing, we may assume that $\delta_1 = \delta_2 = 0$. Thus S_t is a solution of the following SDE:

$$dS_t/S_t = \mu I_{(a,b)}(S_t)dt + \sigma I_{(a,b)}(S_t)dB_t \tag{2}$$

In order to derive a valuation formula, we suppose that we have a contingent claim whose value $C(S, t)$ depends only on S and t, and we construct a replicating portfolio consisting of one contingent claim and $-D$ shares of the stock. Then the value of this portfolio is

$$\Pi = C - DS$$

We also assume there is a self-financing strategy, which means that the increment in the value of this portfolio in one time-step is

$$d\Pi = dC - DdS$$

Since C depends only on S and t, we can apply Ito's lemma to C so that

$$\begin{aligned} dC &= \sigma S I_{(a,b)}(S)\frac{\partial C}{\partial S}dB \\ &\quad + \left\{ \mu S I_{(a,b)}(S)\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 I_{(a,b)}(S)\frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t}\right\}dt \end{aligned} \tag{3}$$

Putting Equations 2 and 3 together, we find that

$$\begin{aligned} d\Pi &= \sigma S I_{(a,b)}(S)\left(\frac{\partial C}{\partial S} - D\right)dS \\ &\quad + \left\{ \mu S I_{(a,b)}(S)\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 I_{(a,b)}(S)\frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} - \mu S D I_{(a,b)}(S)\right\}dt \end{aligned}$$

We can eliminate the random component in $d\Pi$ by choosing $D = (\partial C/\partial S)$. Note that D is the value of $(\partial C/\partial S)$ at the start of the time-step dt. This results in a portfolio whose increment is wholly deterministic, i.e.,

$$\mathrm{d}\Pi = \left\{ \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 I_{(a,b)}(S) \frac{\partial^2 C}{\partial S^2} \right\} \mathrm{d}t$$

Since we assume $r = 0$, $\mathrm{d}\Pi = 0$. Therefore,

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 I_{(a,b)}(S) \frac{\partial^2 C}{\partial S^2} = 0 \tag{4}$$

Hence we have the following

Theorem 4

In the intraday case, the value $C(S, t)$ of the contingent claim $Y(S)$ which expires at the end of the day satisfies the following initial boundary value problem for all $(S, t) \in [a, b] \times [0, T]$:

$$\frac{\partial C}{\partial t}(S, t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}(S, t) = 0$$

$$C(S, T) = Y(S)$$

$$C(a, t) = Y(a)$$

$$C(b, t) = Y(b)$$

Remark

Note that the boundary condition in the above theorem conforms to our intuition. Namely, if the stock price S_t hits the upper limit b, i.e., $S_t = b$ at some time, then S_t will be b afterwards. Since S_t does not change from that point on, the zero interest rate condition means that the value $C(S, t)$ at that point must be the same as $Y(b)$, which is exactly the boundary condition in Theorem 4.

3.2. *Multiday valuation*

In Section 2, we have proposed a model stochastic process which is Markov in the intraday period. But its multiday extension cannot be Markov, as shown in Section 2. This means that we cannot hope to have a *single formula* telling us what $C(S, t)$ is. Rather, we must know the starting price $S_{(n-1)T}$ of the nth trading day in order to properly value $C(S, t)$ for $t \in [(n-1)T, nT]$, since the movement range of the stock price S_t in the nth trading day is determined by the starting price of the same day.

This dependence on the starting price forces us to break the problem into two steps. Before we proceed, let us recall the semigroup property of the solution of the Black and Scholes partial differential equation: Namely, let $C(S, t)$ be the value of the contingent claim $Y(S)$ at time $t \in [0, T]$ with the stock price S where T is the expiry time. Then $C(S, 0)$ can be computed as the value at time 0 of the contingent claim $C(S, \tau)$ with the expiry τ. This semigroup property can be used in two steps for our valuation procedure. First, note that the discrete process obtained by restricting the time to the end of each trading day is Markov, and we can compute the value

$C(S, nT)$ of the contingent claim at the end of nth trading day. Second, once $C(S, T)$ is computed, $C(S, t)$ for $0 \leqslant t \leqslant T$ can be computed by solving the initial boundary value problem in Theorem 4.

For the simplicity of notation, we will use the logarithm of the stock price. So for the rest of this section, x, y, or z will be a variable denoting $\log S$. Then the movement range of the stock price x is $[x - a, x + a]$. We also assume that there are N trading days left, including today, until the expiry. We scale the time so that the nth trading day corresponds to the time interval $[(n-1)T, nT]$. We may also assume without loss of generality that we are only interested in valuing the contingent claim for time $t \in [0, T]$.

Let the contingent claim $Y(y)$ at the expiry $t = NT$ be given. Our valuation process is divided into two steps: The first step computes the value $C(y, nT)$ of the contingent claim at the end of each trading day, and the second step computes $C(y, t)$ for $t \in [0, T]$. The process is summarized below.

3.2.2. *The recursive procedure for multiday valuation*

Step I. Computation of $C(y, nT)$.

Suppose $C(z, nT)$ is given for all $z \in [x - na, x + na]$.

Then compute $C(y, (n-1)T)$ for $y \in [x - (n-1)a, x + (n-1)a]$ as follows:

(i) For each $y \in [x - (n-1)a, x + (n-1)a]$, consider the movement range $[y - a, y + a]$ of the single day(nth day), and restrict $C(z, nT)$ to $z \in [y - a, y + a]$. (For the possible movement range, see Fig. 1.)

(ii) Using Theorem 4, we can find $C(y, (n-1)T)$.

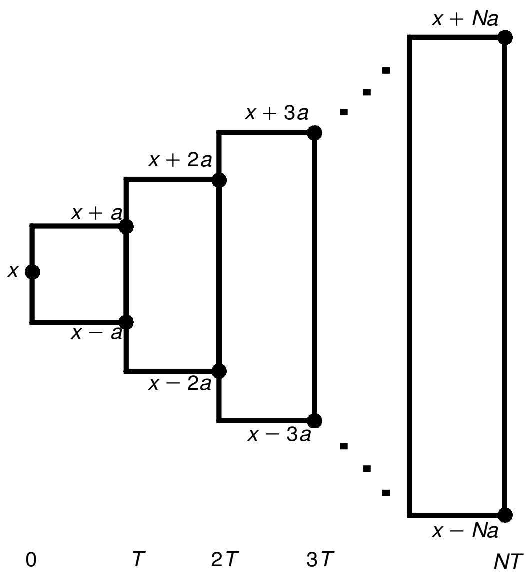

Fig. 1. Possible movement range of $\log S$.

(iii) Using (i) and (ii) above, we can find $C(y, (n-1)T)$ for all $y \in [x-(n-1)a, x+(n-1)a]$.

This way, we can compute $C(y, NT)$, $C(y, (N-1)T)$, $\cdots$, $C(y, T)$.

Step II. Computation of $C(y, t)$.

By Step I, $C(z, T)$ is found for all $z \in [x - na, x + na]$.

Then we can apply Theorem 4 again to compute $C(y, t)$.

Remark

The actual computation must be carried out numerically.

4. Vanishing transaction cost technique

In this section we study the options valuation problem for the situation more general than that considered in Section 3. We assume now that the price process is represented as a solution of Equation 1 for positive δ_1, δ_2, ρ_1 and $\rho 2$. Thus the boundary is slowly reflecting. We also assume that there are no transaction costs, but the interest rate, compounded continuously, is allowed to be nonzero. One should notice that in this case, as explained in the Introduction, there are two inherent arbitrage opportunities. That is, when the price reaches the upper limit, by selling short at the upper limit and putting the proceeds in the bank, and then covering it back

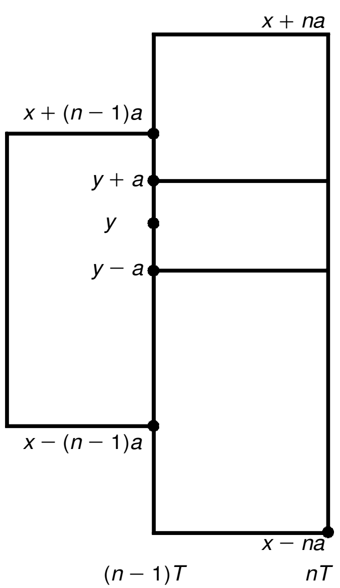

Fig. 2. Valuation of contingent claim at y in $[x-(n-1)a, x+(n-1)a]$ at $t=(n-1)T$.

at a lower price at the end of the trading day, the trader may get profits from the decline of the stock price, and he or she also get the sure interest gain.

The presence of arbitrage causes severe problem in the options valuation. In fact, there is not much one can do when one adheres to the framework of the standard arbitrage pricing theory because all the known methods must presuppose the no arbitrage condition. However, when we ponder on the practicality of the aforementioned arbitrage opportunities, it is natural to realize that they cannot be profitably employed in practice largely because of the transaction costs, which suggests that the transaction costs must play some role. In this section, we devise a technique of incorporating this idea: Namely, we discretize the time interval and use the model with the transaction costs in the style of Leland (1985), and then we let the transaction costs vanish sufficiently fast as the interval of discretization gets to zero. This way, the transaction costs disappear in the limit, but the 'residual effect' of the presence of transaction costs lingers. As a result, we get a parabolic initial boundary value problem. This approach is the same as the one Wilmott *et al*. (1993) used in deriving a partial differential equation when there are transaction costs. The difference in our case is that we let the transaction costs vanish fast enough so that the nonlinear term in the equation disappears as the interval of discretization becomes zero. This is somewhat similar to obtaining a solution of the Euler equation by artificially adding the viscosity term in the equation and letting the viscosity get to zero. This kind of solution is called a vanishing viscosity solution in fluid mechanics, and in this spirit, we also call our method a 'vanishing transaction cost technique.'

Next, we justify that the valuation obtained in this way can be interpreted as representing the 'value' of the contingent claim. To be more specific, we form a portfolio consisting of stocks and bonds rebalanced with the delta hedging method using the solution of the above mentioned initial boundary value problem. We then show that this portfolio minus transaction costs *replicates* the contingent claim in the sense that the hedging error vanishes almost surely as the interval of discretization becomes zero.

This section is organized as the previous section. We first describe the intraday valuation method, which results in a certain initial boundary value problem of a linear parabolic partial differential equation. The multiday valuation works exactly the same as in Section 3.

4.1. *Intraday valuation*

Assume that a stock price in a single day is represented as the solution of the following stochastic differential equation

$$
\begin{aligned}
&\mathrm{d}S_t/S_t = \mu I_{(a,b)}(S_t)\mathrm{d}t + \sigma I_{(a,b)}(S_t)\mathrm{d}B_t + \delta_1 \mathrm{d}\phi_t - \delta_2 \mathrm{d}\psi_t \\
&I_{\{a\}}\mathrm{d}t = \rho_1 \mathrm{d}\phi_t \\
&I_{\{b\}}\mathrm{d}t = \rho_2 \mathrm{d}\psi_t
\end{aligned}
$$

which is introduced in Section 2. Recall $\delta_i > 0$ indicates that a boundary is not absorbing and ρ_i represents the rate of sojourn of S_t on the boundary. We assume that $\rho_i > 0$, which means that a boundary is not instantaneously reflecting.

We first derive a Leland type result. In other words, we'll find a hedging strategy that make the hedging error tends to zero as $\Delta t \longrightarrow 0$. To handle the local times as $\Delta t \longrightarrow 0$, we write the stock price process as follows.

$$\frac{dS}{S} = \mu I_{(a,b)}(S)dt + \sigma I_{(a,b)}(S)dB_t + \frac{\delta_1}{\rho_1} I_{\{a\}}(S)dt - \frac{\delta_2}{\rho_2} I_{\{b\}}(S)dt \tag{6}$$

Then, over a small interval Δt, this process satisfies the following discrete equation

$$\frac{\Delta S}{S} = \mu I_{(a,b)}(S)\Delta t + \sigma I_{(a,b)}(S)\Phi(\Delta t)^{\frac{1}{2}} + \frac{\delta_1}{\rho_1} I_{\{a\}}(S)\Delta t - \frac{\delta_2}{\rho_2} I_{\{b\}}(S)\Delta t + O(\Delta t^{3/2})$$

where Φ is a normally distributed random variable with mean zero and variance one. Here, the expression $O(\Delta t^{3/2})$ has the following meaning; the random variable X is said to be $O(\Delta t^{3/2})$ if $\limsup/_{\Delta t \to 0} E(X)/\Delta t^{3/2}$ is bounded by some constant C independent of Δt. In general, it can be shown that for a process Z satisfying $dZ/Z = \eta dt + \tau dB_t$

$$\begin{aligned}
\log \frac{Z(t+\Delta t)}{Z(t)} &= \log Z(t+\Delta t) - \log Z(t) \\
&= \int_t^{t+\Delta t} \eta dt + \tau dB_t - \frac{1}{2}\int_t^{t+\Delta t} \tau^2 dt \\
&= \eta \Delta t + \tau\Phi(\Delta t)^{\frac{1}{2}} - \frac{\tau^2}{2}\Delta t
\end{aligned}$$

and $Z(t+\Delta t)/Z(t) = 1 + \eta\Delta t + \tau\Phi(\Delta t)^{\frac{1}{2}} + O(\Delta t^{3/2})$ by Taylor's theorem. This can be easily checked since every moment of Φ is finite and η, τ are constants. Let k represent the rate of transaction cost which is proportional to the value of assets traded, and we can also assume that k vanishes sufficiently fast as Δt becomes zero. Precisely, we assume k is $O(\Delta t)$.

Although we have not so far justified the concept of 'value' of the contingent claim in this setting, let us nonetheless let C be its 'value.' Our job is to justify in what sense the word 'value' can be used, and if so, to derive an equation which the 'value' satisfies. As is the case with the delta hedging in the usual use of the Black–Scholes formula, let P be a portfolio consisting of $N = (\partial C/\partial S)$ shares of stocks and $B = C - (\partial C/\partial S)S$ dollars of risk-free security over the interval Δt.

Over the interval Δt, the return of portfolio P will be

$$\begin{aligned}
\Delta P &= N\Delta S + Br\Delta t + O(\Delta t^2) \\
&= C_S S\left(\frac{\Delta S}{S}\right) + (C - C_S S)r\Delta t + O(\Delta t^2)
\end{aligned}$$

where the term $O(\Delta t^2)$ comes from the continuous compounding of interest. The change in the value of the contingent claim C will be

$$\begin{aligned}
\Delta C &= C(S+\Delta S,\, t+\Delta t) - C(S,\, t) \\
&= C_S S\left(\frac{\Delta S}{S}\right) + \frac{1}{2}C_{SS}S^2\left(\frac{\Delta S}{S}\right)^2 + C_t\Delta t + O(\Delta t^{3/2})
\end{aligned}$$

Note that the expression $O(\Delta t^{3/2})$ is written as in the equation for $\Delta S/S$ using the same argument together with the fact that the price movement is bounded. The hedging error ΔH over the interval Δt is

$$\Delta H = \Delta P - \Delta C - k|(S + \Delta S)\Delta N|$$

Apply Taylor's theorem for small ΔS and Δt to have

$$\begin{aligned} k|(S+\Delta S)\Delta N| &= k|(S+\Delta S)(C_S(S+\Delta S,\, t+\Delta t) - C_S(S,\, t))| \\ &= k|(S+\Delta S)C_{SS}(S,\, t)\Delta S| + O(\Delta t^{3/2}) \\ &= k\left|C_{SS}S^2\frac{\Delta S}{S}\right| + O(\Delta t^{3/2}) \end{aligned}$$

Then,

$$\Delta H = (C - C_S S)r\Delta t - \frac{1}{2}C_{SS}S^2\left(\frac{\Delta S}{S}\right)^2 - C_t\Delta t - k\left|C_{SS}S^2\frac{\Delta S}{S}\right| + O(\Delta t^{3/2})$$

Substituting for $\Delta S/S$ gives

$$\Delta H = (C - C_S S)r\Delta t - \frac{1}{2}C_{SS}S^2\sigma^2 I_{(a,b)}\Phi^2\Delta t - C_t\Delta t - k|C_{SS}S^2\sigma I_{(a,b)}\Phi(\Delta t)^{\frac{1}{2}}| + O(\Delta t^{3/2})$$

Observe that the terms which represent the boundary conditions become $O(\Delta t^2)$ due to the fact that $k = O(\Delta t)$. Therefore, taking expectation we have

$$\begin{aligned} E(\Delta H) = &\left((C - C_S S)r - \frac{1}{2}C_{SS}S^2\sigma^2 I_{(a,b)}(S) - C_t - \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{k}{(\Delta t)^{\frac{1}{2}}}|C_{SS}S^2|\sigma I_{(a,b)}(S)\right)\Delta t \\ &+ O(\Delta t^{3/2}) \end{aligned}$$

Now take $\Delta t \longrightarrow 0$. Then the last term in the above equation vanishes as $k = O(\Delta t)$. We require C to satisfy

$$(C - C_S S)r - \frac{1}{2}C_{SS}S^2\sigma^2 I_{(a,b)}(S) - C_t = 0$$

Then, following the argument of Wilmott *et al.* (1993), we can conclude that the expectation of the infinitesimal hedging error becomes zero. The above equation can be rewritten as

$$\left.\begin{aligned} &\frac{\partial C}{\partial t}(S,\, t) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}(S,\, t) + rS\frac{\partial C}{\partial S}(S,\, t) - rC(S,\, t) = 0 \\ &C(S,\, T) = Y(S) \\ &\frac{\partial C}{\partial t}(a,\, t) + ra\frac{\partial C}{\partial S}(a,\, t) - rC(a,\, t) = 0 \\ &\frac{\partial C}{\partial t}(b,\, t) + rb\frac{\partial C}{\partial S}(b,\, t) - rC(b,\, t) = 0 \end{aligned}\right\} \qquad \text{(IBP)}$$

for $(S, t) \in [a, b] \times [0, T]$.

The derivation of (IBP) above is a heuristic one. We now justify that the solution of the above (IBP) can be indeed used as representing the 'value' of the contingent claim. Namely, we have the following result:

Theorem 5

Let C be the solution of (IBP) above, and let P be a portfolio which, over the interval Δt, consists of C_S shares of stocks and $C - C_S S$ dollars of risk-free security. If the transaction cost rate k is assumed to vanish sufficiently fast as Δt becomes to zero, e.g., $k = O(\Delta t)$. Then the hedging error in replicating the contingent claim with this security together with the transaction costs vanishes almost surely as $\Delta t \to 0$. In this sense, it is justifiable to say that the solution C represents the 'value' of the contingent claim.

Proof

Since C satisfies (IBP), we have

$$E(\Delta H) = O(\Delta t^{3/2})$$

Since $E(\Delta H^2/\Delta t^2) < C$ for some constant C, the law of large numbers, referred to Feller (1971), implies

$$\sum_{t=0}^{T-\Delta t} \Delta H_t \to 0 \quad \text{a.s.}$$

where ΔH_t is the hedging error over $[t, t + \Delta t]$ and T is the time to maturity. Therefore, the hedging error over the period $[0, T]$ vanishes almost surely as $\Delta t \to 0$.

4.2. *Multiday valuation*

The multiday valuation in this case works exactly the same as given in Section 3.2. The only difference is that in this case (IBP) above is used instead of the initial boundary value problem in Theorem 4.

5. Conclusion

The market with daily price limit presents a challenging problem in the theory of option valuation mainly because of the presence of the arbitrage opportunities. In the normal application of the arbitrage pricing theory, this makes the options valuation impossible. However, adopting the technique which we call the 'vanishing transaction cost technique,' of introducing transaction cost for a discretized problem and letting the transaction cost vanish sufficiently fast as the size of the discretization interval shrinks to zero, we were able to devise a scheme of successively applying the solution of a certain initial boundary value problem for a parabolic partial differential equation. We then justified this valuation result as representing the 'value' of the contingent claim in the sense that the hedging error becomes zero as the size of the discretization interval shrinks to zero.

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